Vector Algebra

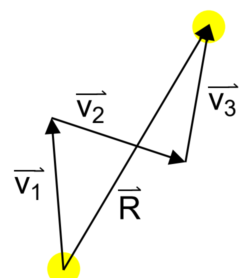
Vectors are representations of numbers that include a **direction**. In vectors, the number part is called the **magnitude**. Some things like velocity, force or field strength are properly measured as vectors, since both the direction and magnitude are needed to fully describe the situation. A force is both the strength of the force and the direction the force is going. Stating a numerical value only does not give a complete picture.

Other measurement values, like time, energy or temperature, don't have meaningful directions. The quantities are called **scalars**.

In previous physics courses, you added vectors and resolved them into components. This worksheet will review these processes and examine a more mathematical way of working with vectors by relating them to a coordinate system.

THE GEOMETRIC APPROACH

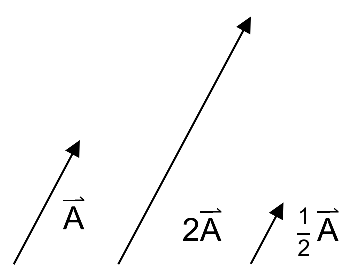
Vectors are drawn as arrows, with the length of the arrow representing the magnitude of the vector. If we have two vectors that represent, say, forces that are acting on the same object, the tails of the two vectors should be on the object in a diagram. On the other hand, vectors are added tip to tail. That means that to draw vectors that we're adding, the tail end of the second vector should start from the arrowhead of the first vector. Vectors can be moved around — they have magnitude and direction, but not *position*. Where we draw the vectors is irrelevant, so long as the size and direction of the vector is maintained. The sum of a series of vectors is drawn as the arrow with its tail at the location of the tail of the first vector, and its head at the head of the last vector. A vector that is the sum of other vectors is often called the **resultant**.



$\vec{v_1} + \vec{v_2} + \vec{v_3} = \vec{R}$

We don't ever really subtract vectors. A calculation like $\vec{A} - \vec{B}$ really means $\vec{A} + (-\vec{B})$, where $-\vec{B}$ is the vector that has the same magnitude as $\vec{B}$, but the opposite direction. In a diagram, it's $\vec{B}$ with the arrowhead moved to the other end of the line.

We can multiply a vector by a real number (i.e., not another vector) to alter its magnitude. A vector obtained this way is called a **scalar multiple** of the first vector. So $2\vec{A}$ is a vector in the same direction as $\vec{A}$, but its magnitude is double. Scalar multiplication can never change the direction of a vector.



If we add $\vec{A} + (-\vec{A})$ the head and tail of the resultant are at the same point. This vector is the **zero vector**, $\vec{0}$. The zero vector has a magnitude of 0, and its direction is undefined. We also get the zero vector by scalar-multiplying any

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vector by 0: $0\vec{A} = \vec{0}$.

ALGEBRAIC PROPERTIES OF VECTOR ADDITION & SCALAR MULTIPLICATION

Just as we can describe properties of operations on real numbers, there are rules to describe adding vectors. If $\vec{A}$, $\vec{B}$ and $\vec{C}$ are three vectors:

$\vec{A} + \vec{B} = \vec{B} + \vec{A}$ *(Vector addition is commutative.)*
$\vec{A} + (\vec{B} + \vec{C}) = (\vec{A} + \vec{B}) + \vec{C}$ *(It's associative.)*
$\vec{A} + \vec{0} = \vec{A}$ *(It has an additive identity.)*
For any $\vec{A}$, $\vec{A} + (-\vec{A}) = \vec{0}$ *(It has additive inverses.)*
For any $\vec{A}$, $\vec{B}$: $\vec{A} + \vec{B}$ is a vector. *(Vector addition is closed.)*

…and for scalar multiplication, if m and n are any real numbers:

For any m, and $\vec{A}$: $m\vec{A}$ is a vector. *(Scalar multiplication is closed.)*
$m\vec{A} + n\vec{A} = (m + n)\vec{A}$ *(The distributive law holds for both the scalars*
$m\vec{A} + m\vec{B} = m(\vec{A} + \vec{B})$ *and the vectors.)*
$(-m)\vec{A} = m(-\vec{A})$
$m(n\vec{A}) = (mn)\vec{A}$
$m\vec{A} + (-m\vec{A}) = \vec{0}$

THE UNIT VECTORS

We define two vectors, each with a magnitude of 1, called **unit vectors**. One points in the positive x-direction (which in this course is called **î** or "i-hat") and one points in the positive y-direction (which we call **ĵ**). For any vector we can draw, we can resolve it into an x-component and a y-component. The x-component has to be a scalar multiple of î and so we can represent it as pî, where p is a real number. Similarly, the y-component can be written as a scalar multiple of ĵ, or qĵ, where q is a real number. Since these components can be added together to get our original vector as the resultant, we can represent any vector $\vec{A}$ as a sum of two vectors, pî + qĵ. If we express vectors in terms of these unit vectors, they form a coordinate system similar to that of a graph, and we can more easily do vector addition and scalar multiplication by just manipulating the coefficients on î and ĵ:

$$(p_1\hat{i} + q_1\hat{j}) \pm (p_2\hat{i} + q_2\hat{j}) = (p_1 \pm p_2)\hat{i} + (q_1 \pm q_2)\hat{j}$$
$$m(p_1\hat{i} + q_1\hat{j}) = mp_1\hat{i} + mq_1\hat{j}$$

NOTES

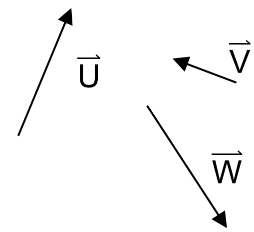
Although î and ĵ are vectors, they don't get the usual arrow symbol over them in the Physics 1 textbook. Other sources do use arrows.

It is possible to multiply vectors with each other; you'll learn about it later in Physics 2.

EXERCISES

A. Sketch the following results of vector addition and scalar multiplication of the vectors $\vec{U}$, $\vec{V}$, and $\vec{W}$ shown below.

1) $\vec{U} + \vec{V}$
2) $\vec{W} - \vec{V}$
3) $2\vec{U}$
4) $-\frac{1}{4}\vec{W}$
5) $2\vec{U} + \vec{W}$
6) $3\vec{V} - \frac{1}{2}\vec{U}$



B. Simplify.

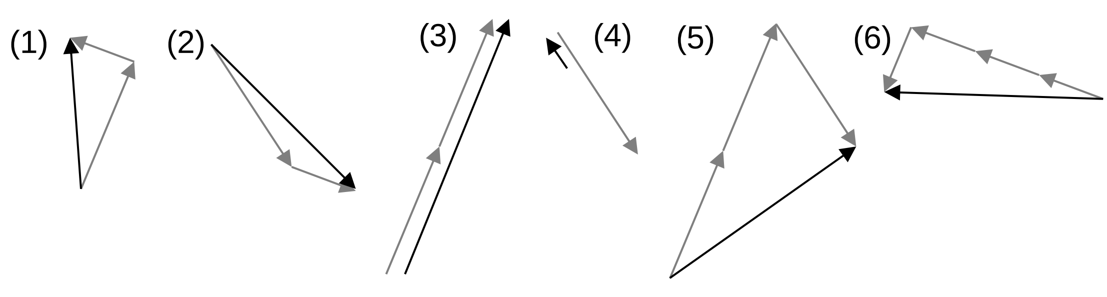
1) $5\vec{A} - 3\vec{A}$
2) $4(2\vec{B})$
3) $9(\vec{C} + \vec{D})$
4) $7(-\vec{E})$
5) $6(3\vec{F} - 2\vec{G}) + 3(4\vec{G} - 6\vec{H})$
6) $2\vec{K} - \vec{0}$
7) $37\vec{L} - 52\vec{L} + 15\vec{L}$

C. Express the following vector quantities in terms of the unit vectors, if $\vec{P} = (2\hat{\imath} - 5\hat{\jmath})$, $\vec{Q} = (-3\hat{\imath} - 6\hat{\jmath})$ and $\vec{R} = (\hat{\imath} + 7\hat{\jmath})$.

1) $3\vec{R}$
2) $\frac{1}{3}\vec{Q}$
3) $4(\vec{P} - \vec{Q})$
4) $\vec{P} - 2\vec{Q} - \vec{R}$
5) $4(\vec{R} + 2\vec{Q}) - 5(\vec{P} - \vec{R})$
6) $\frac{1}{13}(7\vec{P} - \frac{2}{3}\vec{Q} + 10\vec{R})$

SOLUTIONS

A:



B: (1) $2\vec{A}$ (2) $8\vec{B}$ (3) $9\vec{C} + 9\vec{D}$ (4) $-7\vec{E}$ (5) $18\vec{F} - 18\vec{H}$ (6) $2\vec{K}$ (7) $\vec{0}$

C: (1) $(3\hat{\imath} + 21\hat{\jmath})$ (2) $(-\hat{\imath} - 2\hat{\jmath})$ (3) $(20\hat{\imath} + 4\hat{\jmath})$ (4) $(7\hat{\imath})$ (5) $(-25\hat{\imath} + 40\hat{\jmath})$ (6) $(2\hat{\imath} + 3\hat{\jmath})$