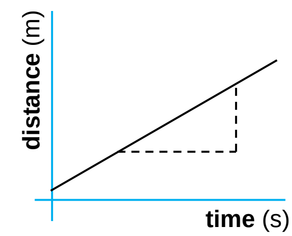
Calculus Concepts for Physics 1

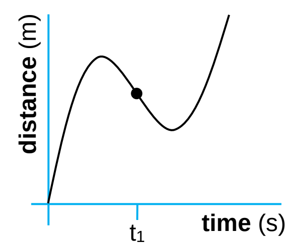
When analyzing motion graphs in university-level physics, we use ideas from calculus to get more information from the graphs. Calculus is a co-requisite, but we encounter concepts from Calculus 1 (and Calculus 2!) much sooner in Physics 1 than we do in those courses. This worksheet will help you with the concepts you need for physics.

DERIVATIVES AND RATES OF CHANGE

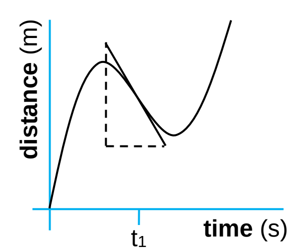
Consider a graph of distance against time:



This motion graph shows an object moving steadily away from the origin. To calculate the object's velocity, we can use the formula $v = \frac{\Delta s}{\Delta t}$, or velocity equals change in position over change in time. Notice that as in all motion graphs time is on the horizontal axis. Distance is on the vertical axis. We can take two points on the graph and do this calculation, and it's exactly the same as performing a slope calculation on those points on the graph by calculating the change in the vertical direction divided by the change in the horizontal direction. The slope of a distance-time graph is the velocity. (Likewise, the slope of a velocity-time graph is acceleration.) With a nice linear graph like this, the calculation is simple, but most examples of motion don't result in straight lines for distance-time graphs.



What if we wanted to know the velocity of the object that produced the graph at the right at time t_1? The graph is not a straight line. We could pick two points, one before t_1 and one after, and do a slope calculation, but it wouldn't describe what happened at t_1 — it would describe the average velocity over the interval between the points we chose. We want the **instantaneous velocity** at the point in time t_1. The basic slope calculation fails us because it operates over an interval. If we ignore this problem and use the point at t_1 for both the initial and final values for the calculation, we get $\frac{0}{0}$, which is indeterminate. We need to find another way to calculate that value.



Calculus gives us a way to cheat. We can't use t_1 for both values, but we can use a t_2 that's a few microseconds before or after t_1, and as long as the motion in the graph is smooth, we'll get a velocity that's close to the right answer. The closer t_1 and t_2 are, the more accurate our slope calculation becomes.

In calculus, we can examine the trend of the slope values as the interval between t_1 and t_2 approaches zero. This is known as the **derivative** of the curve at t_1. It represents the rate of change of the y-values as the x-values increase. For these graphs, that means the slope represents the rate at which position changes as time goes on. That's velocity.

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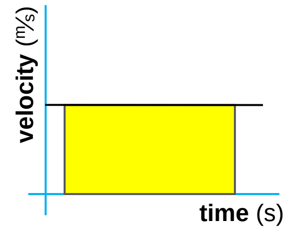
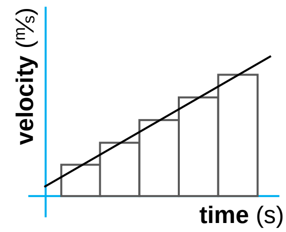
INTEGRALS AND ACCUMULATED VALUES

If we have a velocity vs. time graph and we look at slope, we get acceleration. How do we find the distance covered?

If the velocity is constant, it's simply velocity multiplied by time. But why is this true? The algebra works, but how does it describe what's happening from a physics standpoint?

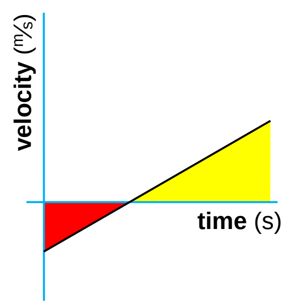


If a vehicle is moving at 10 m/s, then during every second the vehicle moves forward by 10 m. If we're considering the motion of the vehicle over a 5-second interval, then the total distance is 5 × 10 m = 50 m. In terms of the graph, we're multiplying the value on the x-axis by the value on the y-axis. If we look at the area between the graph and the x-axis on that interval, it's a rectangular shape, and multiplying its length (the time) by the width (the velocity), we're calculating the area of the shape.



What if the vehicle is moving with constant acceleration? A vehicle that starts at 4 m/s and accelerates consistently at 2 m/s² will go about 4 m during the first second, 6 m during the next second, 8 m during the next second after that, and so on. These are approximations, but they're close. Essentially, we're slicing the space under the graph into rectangles. The area of each rectangle represents the distance travelled in that interval of time, just as it did in the first example. The total distance, then, is the accumulated area of all the rectangles. For a straight line graph, the area can also be calculated as a triangle sitting on a rectangle, but if the velocity graph is a curve, this slicing method is the only one we have, and the method gets more exact with smaller intervals (and therefore, narrower rectangles).

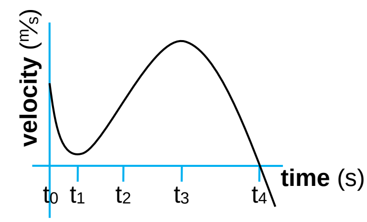
In calculus, we can examine the trend of the total area under the graph as the interval width approaches zero. This is known as the **integral** of the curve over the interval. You'll learn more about these in Calculus 2. For now, you'll only work with straight-line graphs. In a velocity vs. time graph, the area under the curve is displacement; in an acceleration vs. time graph, the area under the curve is velocity. Since taking a derivative and taking an integral are inverse operations (i.e., doing one undoes the other, like multiplying and dividing), any two measurements that share these relationships in one direction will also share them in the other direction.



There's one final concept related to integrals and areas under curves that is important to know, but a little strange given what you've learned in math so far. You know by now that if a velocity curve goes below the x-axis the velocity vector goes to the left. What does this imply about the distance covered? We've been describing displacement as the area under the curve, meaning the area between the curve and the x-axis. If the curve is under the x-axis, then the area between the curve and the axis for that interval counts as "negative area" and subtracts from the total distance. (In the graph here, the velocity starts out negative — to the left — the object reaches a turning point and velocity becomes positive — to the right. Because the object turns around, its displacement decreases.)

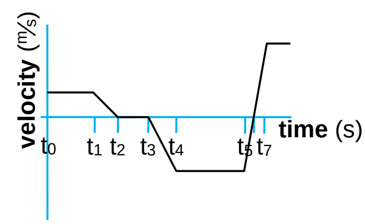
EXERCISES

A. For the velocity vs. time graph at the right, identify the marked times (t_0 – t_4) when:



1) the slope of the curve is positive.
2) the slope of the curve is negative.
3) the slope of the curve is zero.
4) the object is accelerating to the left.
5) the object is accelerating to the right.
6) the object turns around.

B. For the velocity vs. time graph at the right, identify the intervals when:



1) the object is standing still.
2) the object is moving to the left.
3) the object is moving to the right.

Draw a representation of the (4) acceleration vs. time graph, and the (5) distance vs. time graph for this object.

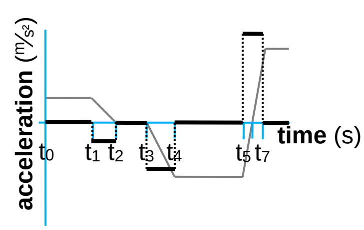
C. A water tank has a drain at the bottom of it. The shape of the tank is such that the flow rate out of the tank steadily gets faster as it drains. A graph of the flow rate against time is a straight line from 0 L/s at t = 0 s to 10 L/s at t = 30 s.

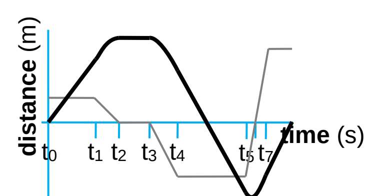
1) Draw a graph of the flow rate of the tank while it drains.
2) What real-world events likely happened in the tank at t = 0 s and t = 30 s?
3) Calculate the area under the graph.
4) What real-world measurement does this area represent?
5) The flow rate changes as time goes on. Does the rate of this change increase, decrease, or stay the same over this period?
6) Determine the rate of change described in (5) for t = 10 s.

SOLUTIONS

A: (1) t_2 (2) t_0, t_4 (3) t_1, t_3 (4) t_0, t_4 (5) t_2 (6) t_4

B: (1) t_2 – t_3 (2) t_3 – t_6 (3) t_0 – t_2, t_6 onwards (4, 5) See graphs.





C: (1) See graph. (2) At t = 0, the drain was opened. At t = 30, all the water has drained out and the tank is empty. (3) 150 L (4) The amount of water that has drained out of the tank. (5) It stays the same, since the slope of the graph is constant. (6) 3 L/s²