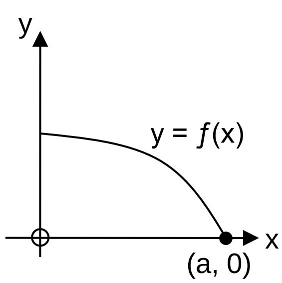
Solids of Revolution

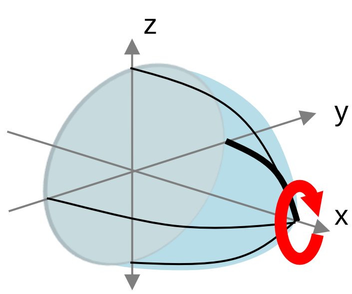
When we rotate a curve around a defined axis, the 3-D shape created is called a **solid of revolution**. In the same way that we can find the area under a curve by calculating the integral of the curve's equation, we can also use integration to calculate the volume of a space swept out (created) by rotating a curve around an axis of rotation. The principle is the same: we chop the volume up into many very thin slices and add up the volumes of the slices to approximate the total, and then use the integral to see what happens with an infinite number of slices.

Since the solid has three dimensions, we have two options: we can slice *perpendicular* or *parallel* to the axis of rotation. If the slices are perpendicular to the axis we make *discs* (or *washers* if the solid has a hole or depression in it), and if the slices are parallel to the axis we make *cylindrical shells*. Since the axis of rotation might be either horizontal or vertical, and the relation that defines the curve might be more easily expressed in terms of x or y, we will decide whether taking perpendicular vs. parallel slices it the best approach for problem solving.

PERPENDICULAR SLICING — DISCS AND WASHERS

We have a continuous curve, bounded in Quadrant I, and it's expressed in terms of $y = f(x)$. We are asked to rotate this curve around the x-axis to form a solid, and determine the volume of this solid. The graph of the curve and the solid are shown below:





The original curve is in bold in the 3-D view. The solid is a bullet shape with a perfectly circular back. To find the volume of the solid, we make slices perpendicular to the x-axis. If the slices are thin enough, they resemble cylinders or discs. The formula for the volume of a cylinder is $\pi \cdot (\text{radius})^2 \cdot (\text{height})$. The height for each disc will be the infinitely small width of a slice, which is represented by "dx". The radius is the distance from the "a" point on the axis of rotation to the point on the curve that has the same x-coordinate, which is $f(x)$. We'll be making slices along the axis of rotation from the start to the finish of the solid. In this case, it starts at origin ($x = 0$) and finishes where the curve intersects the axis ($x = a$). This means the total volume of the slices is $\pi \cdot [f(x)]^2 \cdot dx$ for all values of x from 0 to a.

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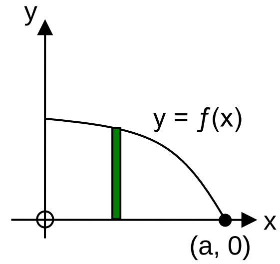
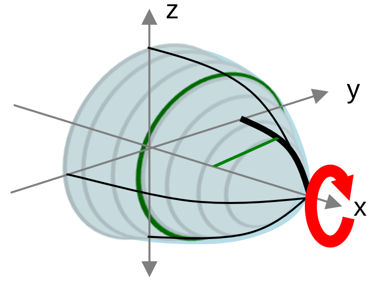
When we cut an infinite number of slices, the sum becomes a Riemann sum of the discs' volumes, which is expressed as an integral:



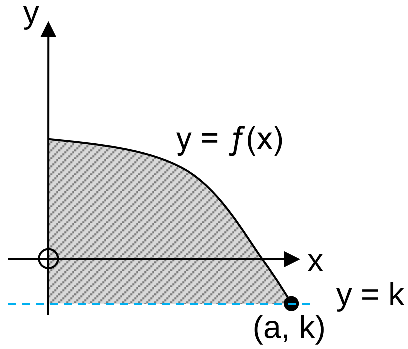


$V = \int_0^a \pi\, [f(x)]^2 = \pi \int_0^a [f(x)]^2.$

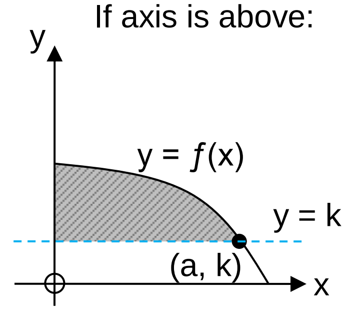
If the axis of rotation isn't the x-axis, but some other line y = k, then we have to adjust the value $f(x)$ to account for the change in radius. We do this using the same method as in algebraic transformations: we're altering the y value, so we subtract k from $f(x)$:

$$V = \pi \int_0^a [f(x) - k]^2\, dx$$

If axis is below:



If axis is above:

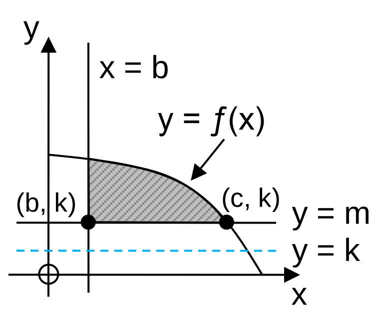


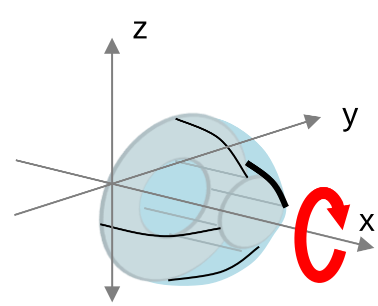
In both cases, the radius of the curve at any point is $f(x) - k$.

Note: $[f(x) - k]^2$ is not the same thing as $f(x)^2 - k^2$.

Now consider the possibility that the curve is not bounded by the axes and the axis of rotation is not the x- or y-axis. If we have the same curve $f(x)$ bounded by the lines y = m and x = b, and rotated around the line y = k, where k < m:

This shape has a cylindrical hole in its centre. We can calculate the volume of this new shape by finding the volume of the shape without the hole, and then subtracting the volume of the hole.





In general, if the function that defines the inside surface of such a shape is $g(x)$, then:

$$V = V_{outside} - V_{inside} = \pi\int_b^c [f(x) - k]^2\, dx - \pi\int_b^c [g(x) - k]^2\, dx$$
$$= \pi \left\{\int_b^c [f(x) - k]^2\, dx - \int_b^c [g(x) - k]^2\, dx \right\}$$

In this particular example, $g(x) = m$, so the inside radius is constant, m − k.

If the axis of rotation has been moved to the other side of the curve as the x- or y-axis (as appropriate), then the adjustment in these formulas is reversed: $[k - f(x)]$.

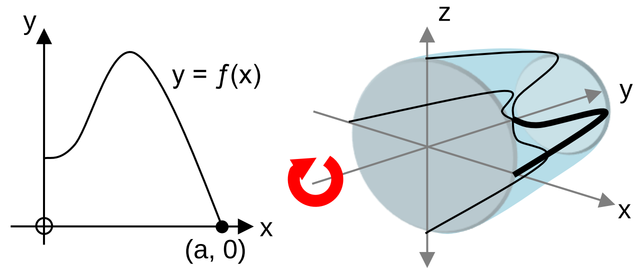
The slices in the Riemann sum associated with this integral are circular with a hole in them, so they are sometimes called **washers**, after the metal part you find on a bolt.

All of these examples might just as easily have been described as a relation $x = f(y)$ and the resulting curves rotated around vertical axes. In this case the x's and y's in the formulas and integrals would be switched. Everything else remains the same.

PARALLEL SLICING — CYLINDRICAL SHELLS

Consider a curve, $y = f(x)$, on an interval [0, a] whose inverse would not be a function. Let's say for simplicity's sake that a is a zero of $f(x)$. The curve is rotated around the y-axis to create a solid of revolution as shown below:

We *could* make slices perpendicular to the axis of rotation — since our axis of rotation is vertical, we'd make horizontal slices and integrate with respect to y — but the inverse relation $f^{-1}(y)$ is inconvenient to work with because it introduces an interior surface for part of the solid. It's usually difficult to separate the interior and exterior sections of the curve to write the integral.



It would be better to integrate with respect to x (using slices parallel to the axis of rotation) so we avoid this problem. Our slices would then be rotated just like the curve was, describing paper-thin cylinders, or **cylindrical shells**. Just like a hollow cylinder made out of a rectangle of paper, we can approximate the volume of the shell by finding the volume of the piece of paper (a rectangular solid) that we would roll up to get the shell. The volume of the shell, then is (length) × (width) × (height). The width of the shell is dx. The height of any given shell is $f(x)$. The length will be the circumference of the shell, which is 2π times the radius. The radius is the x value at the shell's position. The total volume of the solid is the sum of the volumes of all the shells with radii from 0 to a. With an infinite number of slices, we get an integral. The formula becomes:

$$V = \int_0^a 2\pi x \cdot f(x)\,dx = 2\pi \int_0^a x \cdot f(x)\,dx$$

If the axis of rotation isn't the y-axis, but some other arbitrary vertical line, $x = h$, then the height of each shell doesn't change, but the radius does:

$$V = 2\pi \int_b^c (x - h) \cdot f(x)\,dx$$

If the axis has been shifted to the other side (or outside) of the curve, the radius should be expressed as $(h - x)$ instead.

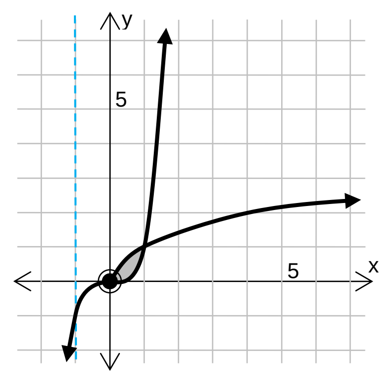
For a relation $x = g(y)$ that's rotated around a horizontal axis, the same strategy applies, and all the x's and y's switch places.

There is no analogue to washers for the cylindrical shells method of finding the volume. We tend to use cylindrical shells to integrate parallel to the axis because the curve isn't a function in the other direction. If there are holes in the direction of integration as well, then the relation isn't a function in any direction!

So which strategy do you use for any given question? There's no firm rule. Sometimes it's much easier to integrate $y \cdot f^{-1}(y)$ than it is $[f(x)]^2$, or vice versa. Sometimes, as in the shape above for cylindrical cells, there's a partial depression created by a curve that overlaps itself in one direction (i.e., it fails the Vertical or Horizontal Line Test, as appropriate); in that case, use the other direction. Use your better judgment, and if slicing in one direction doesn't work, try the other one.

Example 1: Find the volume of the solid of revolution bounded by the curves $y = \sqrt{x}$ and $y = x^3$, and rotated around the axis $x = -1$.

Solution: First we sketch the graph and find the points of intersection. The graph is at right, and we get the intersection points (0, 0) and (1, 1) by inspection.



Which system should we use? Since these are both simple power functions, and they're both already written in terms of x, we should be able to use cylindrical shells.

$$V = 2\pi \int_b^c (x - h) \cdot f(x)\, dx$$

The axis of rotation is at $x = -1$, so $h = 1$. The height of the shaded shape at any value of x on [0, 1] is the difference between the y-value for the curve that defines the top of the shape, $y = \sqrt{x}$, and the y-value for the curve that defines the bottom of the shape, $y = x^3$. We work these into the integral equation and evaluate:

$$\begin{aligned} V &= 2\pi \int_b^c (x - h) \cdot f(x)\, dx \\ &= 2\pi \int_0^1 (x + 1) \cdot (\sqrt{x} - x^3)\, dx \\ &= 2\pi \int_0^1 x^{3/2} - x^4 + x^{1/2} - x^3\, dx \\ &= 2\pi \left[\frac{2}{5}x^{5/2} - \frac{1}{5}x^5 + \frac{2}{3}x^{3/2} - \frac{1}{4}x^4\right]_0^1 \\ &= 2\pi \left[\left(\frac{2}{5} - \frac{1}{5} + \frac{2}{3} - \frac{1}{4}\right) - (0 - 0 + 0 - 0)\right] \\ &= \frac{37\pi}{30} \end{aligned}$$

If the question specifically asked that we use washers, we must change each function into its inverse. The curve that defines the outside of the shape (farther from the axis of rotation) becomes $x = \sqrt[3]{y}$ and the curve that defines the inside of the shape (closer to the axis of rotation) becomes $x = y^2$.

$$\begin{aligned} V &= \pi \int_b^c [f(y) - h]^2 - [g(y) - h]^2\, dy \\ &= \pi \int_0^1 (\sqrt[3]{y} + 1)^2 - (y^2 + 1)^2\, dy \end{aligned}$$

$$= \pi \int_0^1 \sqrt[3]{y^2} + 2\sqrt[3]{y} + 1 - y^4 - 2y^2 - 1 \, dy$$
$$= \pi \int_0^1 y^{2/3} + 2y^{1/3} - y^4 - 2y^2 \, dy$$
$$= \pi \left[\frac{3}{5}y^{5/3} + \frac{3}{2}y^{4/3} - \frac{1}{5}y^5 - \frac{2}{3}y^3 \right]_0^1$$
$$= \pi \left[\left(\frac{3}{5} + \frac{3}{2} - \frac{1}{5} - \frac{2}{3} \right) - (0 - 0 + 0 - 0) \right]$$
$$= \frac{37\pi}{30}$$

The answers agree, which is what we would expect.

EXERCISES

A. Find the volume of the solid of revolution formed by rotating the region bounded by the listed curves about the given axis. Use the method of discs or washers.

1) $2y + 3x = 12$, $y = 0$, $x = 0$; about the y-axis
2) $y = \ln x$, $x = 0$, $y = 0$, $y = 3$; about the y-axis
3) $y = \sec x$ where $x \in (^{-\pi}\!/_2, {}^{\pi}\!/_2)$, $y = 2$; about the x-axis
4) $y = 2 - {}^1\!/_x$, $y = 0$, $x = 2$; about $x = 2$
5) $y = x^2$, $y = -{}^1\!/_2 x^2 + {}^3\!/_2$; about $y = 3$

B. Find the volume of the solid of revolution formed by rotating the region bounded by the listed curves about the given axis. Use the method of cylindrical shells.

1) $y = x^3$, $y = 2x + 4$, $x = 0$; about the y-axis
2) $y = \sqrt{x}$, $y = {}^1\!/_2 x - 4$, $x = 0$; about the x-axis
3) $x = 4 - y^2$, $y = 2 - x$; about $y = 5$
4) $y = \sqrt{1 + x^2}$, $y = 0$, $x = 0$, $x = 1$; about the y-axis
5) $y = \frac{1}{x^2 - 3x}$ on $x \in [{}^3\!/_2, \infty)$, $y = -2$, $x = {}^3\!/_2$; about $x = {}^3\!/_2$

C. Find the volume of the solid of revolution formed by rotating the region bounded by the listed curves about the given axis. Use whichever method is more appropriate.

1) $y = -4x^2 + 8x$, $y = 0$; about the x-axis
2) $x^2 + (y - 4)^2 = 9$, $x = 0$, $y = 4$; about the y-axis
3) $y = \sin x^2$, $x \in [0, 2)$, $y = 0$; about the y-axis

D. 1) Find the volume of the solid of revolution bounded by the curves $y = 3x^3 - 15x^2 + 20x$ and $y = 2x$ on the interval $[0, 2]$, rotated about the y-axis, using the method of cylindrical shells.

2) Do the same question, but on the interval [0, 3]. Compare this answer to D1. Is there anything strange?

3) Find the correct volume for the solid on the interval [0, 3] by splitting the integral in an appropriate place.

4) Find the volume of the solid of revolution bounded by the curves $y = (x + 1)^2$ and $y = x^3 + 1$ rotated about $x = 2$.

SOLUTIONS

A: (1) 32π (2) $\frac{\pi}{2}(e^6 - 1)$ (3) $\frac{8\pi^2}{3} - 2\pi\sqrt{3}$ (4) $\pi(\frac{15}{2} - 8 \ln 2)$ (5) $\frac{44\pi}{5}$

B: (1) $\frac{208\pi}{15}$ (2) $\frac{256\pi}{3}$ (3) $\frac{81\pi}{2}$ (4) $\frac{\pi}{3}(4\sqrt{2} - 2)$ (5) $\pi(\ln \frac{2}{9} + \frac{7}{2})$

C: (1) discs: $\frac{256\pi}{15}$ (2) either method: 18π (3) cylindrical shells: 2π

D: (1) $\frac{72\pi}{5}$ (2) $\frac{81\pi}{10}$; The volume has decreased. This happened because the curves intersected at x = 2. The volume included in D2 was all negative. (3) $\frac{207\pi}{10}$ (4) $\frac{69\pi}{10}$