Integration by Parts

Integration by parts is a way of using the Product Rule in reverse. It is used in situations where the integrand consists of two (or more) things multiplied together. The integrand is assumed to be one of the two terms that result from an application of the Product Rule. The formula for integration by parts is:

$$\int u\,dv = u \cdot v - \int v\,du$$

One of the two factors in the integrand is integrated, and one is differentiated, and it's up to you to decide which should be which. Notice that the result of integration by parts still has an integral in it. The idea is that this new integral should be simpler to solve than the original one. When choosing which factor (u) will be differentiated, and which factor (dv) will be integrated, remember **DETAIL**. Choose the part that is closer to the D for dv, and the part that is further away for u. This is a rule of thumb — it is a suggestion for what is best, but it doesn't always work perfectly.

dv
Exponential functions
Trigonometric functions
Algebraic functions (such as x^2)
Inverse trigonometric functions
Logarithmic functions
u

Example 1: Find $\int x^3 \ln x\,dx$ using integration by parts.

Solution: We need to split the integrand into two factors, one to integrate, and one to differentiate. Using the chart above, we see that we want ln x to be the *u*, and x^3 to be the *dv* (even though it will actually make that part more complicated, rather than simpler). When calculating v from dv, we don't worry about the "+ c" part:

$$u = \ln x \qquad dv = x^3\,dx$$
$$du = \tfrac{1}{x}\,dx \qquad v = \tfrac{1}{4}x^4$$

$$\begin{aligned}\int x^3 \ln x\,dx &= (\ln x)\cdot(\tfrac{1}{4}x^4) - \int(\tfrac{1}{4}x^4)\cdot(\tfrac{1}{x}\,dx)\\ &= \tfrac{1}{4}x^4\ln x - \tfrac{1}{4}\int x^3\,dx\\ &= \tfrac{1}{4}x^4\ln x - \tfrac{1}{4}\cdot\tfrac{1}{4}x^4 + c\\ &= \tfrac{1}{4}x^4\ln x - \tfrac{1}{16}x^4 + c\end{aligned}$$

Sometimes integration by parts needs to be applied multiple times before the "$\int v\,du$" can be successfully integrated.

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Example 2: Find $\int x^2 e^{2x}\,dx$ using integration by parts.

Solution: The x^2 becomes simpler when differentiated, and more complicated when integrated, while the e^{2x} part stays about the same both ways. Make the x^2 part the u and e^{2x} the dv:

$$u = x^2 \qquad dv = e^{2x}\,dx$$
$$du = 2x\,dx \qquad v = \tfrac{1}{2} e^{2x}$$

$$\int x^2 e^{2x}\,dx = (x^2)\cdot(\tfrac{1}{2} e^{2x}) - \int (2x)\cdot(\tfrac{1}{2} e^{2x}\,dx)$$
$$= \tfrac{1}{2} x^2 e^{2x} - \int x\, e^{2x}\,dx$$

Our result now has a simpler integral, but we still can't evaluate it. We can apply integration by parts again.

$$U = x \qquad dV = e^{2x}\,dx$$
$$dU = dx \qquad V = \tfrac{1}{2} e^{2x}$$

$$= \tfrac{1}{2} x^2 e^{2x} - [\tfrac{1}{2} x\, e^{2x} - \int \tfrac{1}{2} e^{2x}\,dx]$$
$$= \tfrac{1}{2} x^2 e^{2x} - \tfrac{1}{2} x\, e^{2x} + \tfrac{1}{2} \int e^{2x}\,dx$$
$$= \tfrac{1}{2} x^2 e^{2x} - \tfrac{1}{2} x\, e^{2x} + \tfrac{1}{2} (\tfrac{1}{2} e^{2x} + c)$$
$$= \tfrac{2}{4} x^2 e^{2x} - \tfrac{2}{4} x\, e^{2x} + \tfrac{1}{4} e^{2x} + c$$
$$= \frac{2x^2 - 2x + 1}{4} e^{2x} + c$$

Some problems also loop around so that the $\int v\,du$ part is identical to the integral you started with:

Example 3: Find $\int e^{2x} \cos x\,dx$ by using integration by parts.

$$u = e^{2x} \qquad dv = \cos x\,dx$$
$$du = 2e^{2x}\,dx \qquad v = \sin x$$

$$\int e^{2x} \cos x\,dx = e^{2x} \sin x - \int 2e^{2x} \sin x\,dx$$
$$= e^{2x} \sin x - 2 \int e^{2x} \sin x\,dx$$

$$U = e^{2x} \qquad dV = \sin x\,dx$$
$$dU = 2e^{2x}\,dx \qquad V = -\cos x$$

$$= e^{2x} \sin x - 2\,(-e^{2x} \cos x - \int -2e^{2x} \cos x\,dx\,)$$
$$= e^{2x} \sin x + 2\, e^{2x} \cos x - 4 \int e^{2x} \cos x\,dx$$

We've applied integration by parts twice, and the integral that has resulted is the same as the one we started with. Call this integral I, and then solve for I.

$$I = e^{2x} \sin x + 2\, e^{2x} \cos x - 4\, I$$
$$5I = e^{2x} \sin x + 2\, e^{2x} \cos x$$
$$I = \tfrac{1}{5} e^{2x} \sin x + \tfrac{2}{5} e^{2x} \cos x + c$$

EXERCISES

A. Evaluate using integration by parts.

1) $\int x\, e^x\, dx$

2) $\int \tan^{-1} z\, dz$ [*Hint: dv = dz.*]

3) $\int 3^{t+1}\, dt$

4) $\int x \sec^2 x\, dx$

5) $\int x \sec x \tan x\, dx$ [*Hint: u = x.*]

6) $\int (x^2 + x) \sin 4x\, dx$

7) $\int \sin \beta \cos \beta\, d\beta$

8) $\int (3x^2 + 1)(\tan^{-1} x)\, dx$

9) $\int \sin \gamma \sin 3\gamma\, d\gamma$

SOLUTIONS

Other solutions are possible with trig-based integrals. Differentiate your solution to check your answer if it doesn't match these:

A: (1) $(x - 1)\, e^x + c$

(2) $z \tan^{-1} z - \frac{1}{2} \ln (1 + z^2) + c$

(3) $3^{t+1} \div \ln 3$

(4) $x \tan x - \ln |\sec x| + c$

(5) $x \sec x - \ln |\sec x + \tan x| + c$

(6) $\frac{-8x^2-8x+1}{32} \cos 4x + \frac{2x+1}{16} \sin 4x + c$

(7) $-\frac{1}{2} \cos^2 \beta + c$ *or* $\frac{1}{2} \sin^2 \beta + c$

(8) $(x^3 + x)(\tan^{-1} x) - \frac{1}{2}x^2 + c$

(9) $-\frac{3}{10} \sin \gamma \cos 3\gamma + \frac{1}{10} \cos \gamma \sin 3\gamma + c$ *or* $-\frac{3}{8} \sin \gamma \cos 3\gamma + \frac{1}{8} \cos \gamma \sin 3\gamma$