Easy Integrals (That Don't Look Easy)

Sometimes it is possible to simplify a function in an integral before you integrate it, making a difficult problem into a simple one.

Example 1: Evaluate the integrals: a) $\int (x^4-5)^2\,dx$ b) $\int (x^3+x)(2x^2-4)\,dx$

c) $\int \frac{x^3+4x-3}{x}\,dx$ d) $\int \cos x\tan^2 x+\cos x\,dx$ e) $\int \frac{1}{e^{2x}}\,dx$

Solution: a) If there was an x^3 in the integral, we could integrate by parts… but it is much easier to turn this into an ordinary polynomial:

$$\begin{aligned}\int (x^4-5)^2\,dx &= \int (x^4-5)(x^4-5)\,dx\\ &= \int x^8-10x^4+25\,dx\\ &= \tfrac{1}{9}x^9-2x^5+25x+c\end{aligned}$$

b) Same idea again. Just expand:

$$\begin{aligned}\int (x^3+x)(2x^2-4)\,dx &= \int 2x^5-2x^3-4x\,dx\\ &= \tfrac{1}{3}x^6-\tfrac{1}{2}x^4-2x^2+c\end{aligned}$$

c) And the same idea again. Divide the denominator into the numerator and the result will be a polynomial:

$$\begin{aligned}\int \frac{x^3+4x-3}{x}\,dx &= \int \frac{x^3}{x}+\frac{4x}{x}-\frac{3}{x}\,dx\\ &= \int x^2+4-3x^{-1}\,dx\\ &= \tfrac{1}{3}x^3+4x-3\ln|x|+c\end{aligned}$$

d) In this case, we can use trigonometric identities to make the integral easier:

$$\begin{aligned}\int \cos x\tan^2 x+\cos x\,dx &= \int (\cos x)(\tan^2 x+1)\,dx\\ &= \int (\cos x)(\sec^2 x)\,dx\\ &= \int \frac{\cos x}{\cos^2 x}\,dx\\ &= \int \sec x\,dx\\ &= \ln|\sec x+\tan x|+c\end{aligned}$$

e) We can also express fractions with negative exponents:

$$\begin{aligned}\int \frac{1}{e^{2x}}\,dx &= \int e^{-2x}\,dx\\ &= -\tfrac{1}{2}e^{-2x}+c\end{aligned}$$

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EXERCISES

A. Evaluate the integrals:

1) $\int (x+1)^3\ dx$

6) $\int \dfrac{\sin^2 x + \sin x \cos x}{\cos^2 x + \sin x \cos x}\,dx$

2) $\int (x^{15} - x^{12})^2\ dx$

7) $\int \sin x \tan x + \cos x\ dx$

3) $\int (x^{\frac{1}{2}} + x^{-\frac{1}{2}})(x^{\frac{5}{2}} - x^{\frac{3}{2}})\ dx$

8) $\int (\sin x + \cos x + 1)(\sin x + \cos x - 1)\ dx$

4) $\int \dfrac{3x^5 - 5x^3}{2x^2}\ dx$

9) $\int \dfrac{e^x + e^{3x}}{e^{2x}}\,dx$

5) $\int \dfrac{x^2 - 3x - 28}{x^3 - 7x^2}\,dx$

10) $\int (1+e^{3x})\cdot e^2\ dx$

SOLUTIONS

A. (1) $\int x^3 + 3x^2 + 3x + 1\,dx = \frac{1}{4}x^4 + x^3 + \frac{3}{2}x^2 + x + c$

(2) $\int x^{30} - 2x^{27} + x^{24}\ dx = \frac{1}{31}x^{31} - \frac{1}{14}x^{28} + \frac{1}{25}x^{25} + c$ (3) $\int x^3 + x\ dx = \frac{1}{4}x^4 + \frac{1}{2}x^2 + c$

(4) $\int \frac{3}{2}x^3 - \frac{5}{2}x\ dx = \frac{3}{8}x^4 - \frac{5}{4}x^2 + c$ (5) $\int x^{-1} + 4x^{-2}\ dx = \ln|x| - \dfrac{4}{x} + c$

(6) $\int \tan x\ dx = \ln|\sec x| + c$ (7) $\int \sec x\ dx = \ln|\sec x + \tan x| + c$

(8) $\int \sin 2x\ dx = -\frac{1}{2}\cos 2x + c$ (9) $\int e^{-x} + e^x\ dx = -e^{-x} + e^x + c$

(10) $\int e^{3x+2} + e^2\ dx = \frac{1}{3}e^{3x+2} + e^2 x + c$ [*e² is just a constant!*]