Direct Proofs

The Laws of Logic and Inference

By now we've seen that we can take simple, or **primitive**, statements and combine them using operators, just as we can with numbers. The operators for statements are:

conjunction	$\wedge$	"and"	$p \wedge q$ is true only if p is true and q is true.
disjunction	$\vee$	"or"	$p \vee q$ is true if either p is true or q is true.
	$\oplus$	"ex-or"	$p \oplus q$ is true if exactly one of p and q is true.
negation	$\neg$	"not"	$\neg p$ is true if p is false; $\neg p$ is false if p is true.
implication	$\rightarrow$	"implies"	$p \rightarrow q$ is true *unless* p is true and q is false
biconditional	$\leftrightarrow$	"iff"	$p \leftrightarrow q$ is true if p and q are both true or both false.

If we combine primitive statements using these connectors we get a **compound statement**. Compound statements can get pretty complex, so it's useful to have a way of simplifying or rewriting compound statements so that they're more understandable.

The Laws of Logic are a list of ways of changing a compound statement that guarantee that the truth value of the statement is unchanged. The similar rule in algebra is that you can replace any part of an expression with anything else that has the same value:

$$\begin{aligned} & 3x + 4y - 7y \\ = \; & 3x - 3y \end{aligned} \qquad\qquad \begin{aligned} & q \rightarrow (p \wedge p) \\ \Leftrightarrow \; & q \rightarrow p \end{aligned}$$

In algebra we indicate equivalence with the equal sign. In logic we use the double-stroke arrow "$\Leftrightarrow$" to indicate that two distinct statements are equivalent. If we write "$p \leftrightarrow q$", that may be a true statement (if the truth values of p and q match) or a false statement, and we look at the both possibilities. There's no such dispute in the example above — we assert that these lines *must always* have matching truth values.

ARGUMENTS AND INFERENCE

We also want to know whether a series of conditions being met guarantees that a conclusion may be drawn — in math, that's the structure of a theorem. In logic, this structure is called an **argument**: given that a series of **premises** are true, can we deduce that a certain conclusion is true? We use arguments to write **proofs**.

We can use the Laws of Inference to take previous statements in an argument to deduce new ones. We see the analogue in algebra in solving systems of equations:

$$\begin{array}{rl} 3x - 3y & = 20 \\ \underline{-3x + 2y} & \underline{= -16} \\ -y & = 4 \end{array}$$

①	$q \rightarrow p$	Premise
②	q	Premise
③	p	Modus Ponens: 1, 2

In solving the system on the left, we are *not* saying that the last equation is equivalent to the other two! If I know that $3x - 3y = 20$, do I know that $-y = 4$? No — that first equation

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by itself isn't enough to draw that conclusion. On the other hand, we demonstrate by adding the two equations together that if they're both true, then the third equation is true.

For the argument on the right, we cannot say that p is true just because (q → p) is true, or just because q is true; we cannot write "(q → p) ⇔ p" or "q ⇔ p". However, the two statements together *do* allow us to conclude p is true. For proofs, we number statements, and we either mark them as premises — statements that we assume are true without justification — or we indicate which law justifies saying that they follow from one or more previous statements.

Both the Laws of Logic and the Laws of Inference define structures of statements that we may simplify or combine. Modus Ponens also applies to the following argument:

① [[(a ∧ b) → (b ∨ c)] ↔ (a ∧ ¬c)] → [(x ∧ y ∧ w) ∨ ¬[t → (y ∨ ¬z)]]	Premise
② [[(a ∧ b) → (b ∨ c)] ↔ (a ∨ ¬c)]	Premise
③ [(x ∧ y ∧ w) ∨ ¬[t → (y ∨ ¬z)]]	Modus Ponens: 1, 2

Here, "p" and "q" are convoluted compound statements, but the structure is identical: if we know *this* implies *that*, and *this* is true, then we also know *that* is true.

DIRECT PROOF

A **direct proof** begins with a list of premises, and ends with a statement that we wish to assert as a conclusion. A full proof in symbolic logic will show all the steps needed to demonstrate that the conclusion follows from the premises, along with the justification for each step. In the strictest sense, the statements must appear *exactly* as they do in the Laws. The following would be invalid:

① p ∧ q	Premise
② q	Conjuctive Simplification: 1

The Rule of Conjunctive Simplification says that we may deduce p, not q, even though we can see that the rule should work on either half of the statement in ①. Strictly speaking, we should include a line between them: "q ∧ p [Commutativity: 1]". In practice, we can save time by combining two steps where one is trivially simple into one line:

… ② q	Comm., Conj. Simp.: 1

A full symbolic proof looks something like this:

① (p ∧ q) → [p → (s ∧ t)]	Premise
② r ∧ (p ∧ q)	Premise
/∴	s ∨ t
③ p ∧ q	Comm., Conj. Simp.: 2
④ p → (s ∧ t)	Modus Ponens: 1, 3
⑤ p	Conjunctive Simplification: 3
⑥ s ∧ t	Modus Ponens: 4, 5
⑦ s	Conjunctive Simplification: 6
⑧ s ∨ t	Disjunctive Amplification: 7

Some important things to notice:

● The "/∴" line indicates what statement we want as our conclusion. It's often read as "To prove:". Since we have a numbered, justified line in our proof that is an exact copy of this statement, this proof is complete.

● Once a line has been written and justified, future lines may refer to it in their justifications as though it was a premise.

● The Laws of Logic may be used in proofs whenever they are useful. (They always apply, whereas the Laws of Inference are only to be used in proofs.)

LAWS OF LOGIC

For any statements, p, q, r, any tautology T_0 and any contradiction F_0:

$\neg\neg p \Leftrightarrow p$	Double Negation
$\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$ $\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$	De Morgan's
$p \vee q \Leftrightarrow q \vee p$ $p \wedge q \Leftrightarrow q \wedge p$	Commutativity
$(p \vee q) \vee r \Leftrightarrow p \vee (q \vee r)$ $(p \wedge q) \wedge r \Leftrightarrow p \wedge (q \wedge r)$	Associativity
$p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$	Distributivity
$p \vee p \Leftrightarrow p$ $p \wedge p \Leftrightarrow p$	Idempotent
$p \vee F_0 \Leftrightarrow p$ $p \wedge T_0 \Leftrightarrow p$	Identity
$p \vee \neg p \Leftrightarrow T_0$ $p \wedge \neg p \Leftrightarrow F_0$	Inverse
$p \vee T_0 \Leftrightarrow T_0$ $p \wedge F_0 \Leftrightarrow F_0$	Domination
$p \vee (p \wedge q) \Leftrightarrow p$ $p \wedge (p \vee q) \Leftrightarrow p$	Absorption

SUBSTITUTION RULES

$p \rightarrow q \Leftrightarrow \neg p \vee q$	Implication
$p \leftrightarrow q \Leftrightarrow (\neg p \vee q) \wedge (\neg q \vee p)$	Biconditional
$p \oplus q \Leftrightarrow (p \wedge \neg q) \vee (\neg p \wedge q)$ $\Leftrightarrow (p \vee q) \wedge \neg(p \wedge q)$	Exclusive Or

LAWS OF INFERENCE

For any statements p, q, r, s and any contradiction F_0:

p $\underline{p \rightarrow q}$ $\therefore q$	Modus Ponens
$p \rightarrow q$ $\underline{q \rightarrow r}$ $\therefore p \rightarrow r$	Syllogism
$p \rightarrow q$ $\underline{\neg q}$ $\therefore \neg p$	Modus Tollens
p $\underline{q}$ $\therefore p \wedge q$	Conjunction
$p \vee q$ $\underline{\neg p}$ $\therefore q$	Disjunct. Syll.
$\underline{\neg p \rightarrow F_0}$ $\therefore p$	Contradiction
$\underline{p \wedge q}$ $\therefore p$	Conj. Simplif'n
$\underline{p}$ $\therefore p \vee q$	Disj. Amplif'n
$p \wedge q$ $\underline{p \rightarrow (q \rightarrow r)}$ $\therefore r$	Conditional Proof
$p \rightarrow r$ $\underline{q \rightarrow r}$ $\therefore (p \vee q) \rightarrow r$	Proof by Cases
$p \rightarrow q$ $r \rightarrow s$ $\underline{p \vee r}$ $\therefore q \vee s$	Constr. Dilemma
$p \rightarrow q$ $r \rightarrow s$ $\underline{\neg q \vee \neg s}$ $\therefore \neg p \vee \neg r$	Destr. Dilemma

EXERCISES

A. Which Law of Logic or Law of Inference is used or demonstrated each of these paragraphs?

1) A statement in disjunction with a tautology is also a tautology.

2) If I left my umbrella at work, I'd have wet hair now. My hair is dry, so I must have left my umbrella somewhere else.

3) It's true that every newspaper is either published online or it's not.

4) Where are our plane tickets? I don't have them. You must have them.

5) I give every customer a receipt, so if you bought these shoes here yesterday, you got a receipt.

6) If he didn't pass both his tests, then he failed either the midterm or the final.

LAWS OF LOGIC
For any statements, p, q, r, any tautology T_0 and any contradiction F_0:

$\neg\neg p \Leftrightarrow p$	Double Negation
$\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$ $\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$	De Morgan's
$p \vee q \Leftrightarrow q \vee p$ $p \wedge q \Leftrightarrow q \wedge p$	Commutativity
$(p \vee q) \vee r \Leftrightarrow p \vee (q \vee r)$ $(p \wedge q) \wedge r \Leftrightarrow p \wedge (q \wedge r)$	Associativity
$p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$	Distributivity
$p \vee p \Leftrightarrow p$ $p \wedge p \Leftrightarrow p$	Idempotent
$p \vee F_0 \Leftrightarrow p$ $p \wedge T_0 \Leftrightarrow p$	Identity
$p \vee \neg p \Leftrightarrow T_0$ $p \wedge \neg p \Leftrightarrow F_0$	Inverse
$p \vee T_0 \Leftrightarrow T_0$ $p \wedge F_0 \Leftrightarrow F_0$	Domination
$p \vee (p \wedge q) \Leftrightarrow p$ $p \wedge (p \vee q) \Leftrightarrow p$	Absorption

SUBSTITUTION RULES

$p \rightarrow q \Leftrightarrow \neg p \vee q$	Implication
$p \leftrightarrow q \Leftrightarrow (\neg p \vee q) \wedge (\neg q \vee p)$	Biconditional
$p \oplus q \Leftrightarrow (p \wedge \neg q) \vee (\neg p \wedge q)$ $\Leftrightarrow (p \vee q) \wedge \neg(p \wedge q)$	Exclusive Or

LAWS OF INFERENCE
For any statements p, q, r, s and any contradiction F_0:

p $\underline{p \rightarrow q}$ $\therefore q$	Modus Ponens
$p \rightarrow q$ $\underline{q \rightarrow r}$ $\therefore p \rightarrow r$	Syllogism
$p \rightarrow q$ $\underline{\neg q}$ $\therefore \neg p$	Modus Tollens
p $\underline{q}$ $\therefore p \wedge q$	Conjunction
$p \vee q$ $\underline{\neg p}$ $\therefore q$	Disjunct. Syll.
$\underline{\neg p \rightarrow F_0}$ $\therefore p$	Contradiction
$\underline{p \wedge q}$ $\therefore p$	Conj. Simplif'n
$\underline{p}$ $\therefore p \vee q$	Disj. Amplif'n
$p \wedge q$ $\underline{p \rightarrow (q \rightarrow r)}$ $\therefore r$	Conditional Proof
$p \rightarrow r$ $\underline{q \rightarrow r}$ $\therefore (p \vee q) \rightarrow r$	Proof by Cases
$p \rightarrow q$ $r \rightarrow s$ $\underline{p \vee r}$ $\therefore q \vee s$	Constr. Dilemma
$p \rightarrow q$ $r \rightarrow s$ $\underline{\neg q \vee \neg s}$ $\therefore \neg p \vee \neg r$	Destr. Dilemma

B. Write the justifications for each of the steps in the proof. The preceding steps used in the justifications are provided (so that the justification of ② relies on statement 1, etc.).

①	$(p \vee q) \rightarrow [q \rightarrow (r \vee p)]$	Premise
②	$\neg(p \vee q) \vee [q \rightarrow (r \vee p)]$	________________: 1
③	$(\neg p \wedge \neg q) \vee [q \rightarrow (r \vee p)]$	________________: 2
④	$[q \rightarrow (r \vee p)] \vee (\neg p \wedge \neg q)$	________________: 3
⑤	$[\neg q \vee (r \vee p)] \vee (\neg p \wedge \neg q)$	________________: 4
⑥	$[\neg q \vee (r \vee p)] \vee (\neg q \wedge \neg p)$	________________: 5
⑦	$\neg q \vee [(r \vee p) \vee (\neg q \wedge \neg p)]$	________________: 6
⑧	$\neg q \vee [(\neg q \wedge \neg p) \vee (r \vee p)]$	________________: 7
⑨	$[\neg q \vee (\neg q \wedge \neg p)] \vee (r \vee p)$	________________: 8
⑩	$\neg q \vee (r \vee p)$	________________: 9
⑪	$q \wedge \neg r$	Premise
⑫	q	________________: 11
⑬	$\neg\neg q$	________________: 12
⑭	$r \vee p$	________________: 10, 12
⑮	$\neg r$	_________, ___________: 11
⑯	p	________________: 14, 15

C. Write the justifications for each of the steps in this version of the same proof. (The preceding steps used in the justifications are provided.)

①	$(p \vee q) \rightarrow [q \rightarrow (r \vee p)]$	Premise
②	$q \wedge \neg r$	Premise
③	q	________________: 2
④	$p \vee q$	_________, ___________: 3
⑤	$(p \vee q) \wedge q$	________________: 4, 3
⑥	$r \vee p$	________________: 5, 1
⑦	$\neg r$	_________, ___________: 2
⑧	p	________________: 6, 7

D. Construct proofs for the following arguments.

1) $a \to b$
$b \lor \neg c$
$/\therefore\ (a \lor c) \to b$

[*Hint:* One of the Laws of Inference has exactly the structure of the conclusion. Work backwards.]

2) $d \land e$
$e \to (f \lor g)$
$\neg g$
$/\therefore\ f \lor h$

[*Hint:* One Law of Inference has a statement in its conclusion that's not among its premises. Work backwards.]

3) $k \to m$
$/\therefore\ \neg m \to \neg k$

* This result may be useful for later. Call the result "Contrapositive".

[*Hint:* Start with a Substitution Rule.]

4) $n \land (n \lor \neg n)$
$p \to \neg n$
$/\therefore\ \neg p$

[*Hint:* Simplify the first premise first.]

5) $(q \lor r) \leftrightarrow s$
$q \oplus r$
$/\therefore\ s$

[*Hint:* Use Substitution Rules on the premises and simplify.]

6) $(t \lor u) \to (t \lor \neg u)$
u
$/\therefore\ t$

[*Hint:* The second premise is enough to let you use Modus Ponens on the first premise.]

7) $\neg w \lor \neg x \lor y$
$y \lor z$
$z \to (w \lor x)$
$\neg y$
$/\therefore\ w \oplus x$

[*Hint:* The last line of the proof must use a Substitution Rule for "$\oplus$". Can you create a proof that gets you to the second-last line?]

8) $a \lor (b \land c)$
$\neg d \to b$
$\neg(d \land a)$
$/\therefore\ \neg b \lor \neg c$

[*Hint:* You can use the Destructive Dilemma to complete this problem.]

9) $\neg e \to (e \lor f)$
$e \to (f \land g)$
$/\therefore\ f$

[*Hint:* Start by simplifying both premises by Implication. Combine them to get to the conclusion.]

SOLUTIONS

A: (1) Domination (2) Modus Tollens (3) Inverse (4) Disjunctive Syllogism (5) Syllogism (6) DeMorgan's Law

B: (2) Implication (3) DeMorgan's (4) Commutativity (5) Implication (6) Commutativity (7) Associativity (8) Commutativity (9) Associativity (10) Absorption (12) Conjunctive Simplification (13) Double Negation (14) Disjunctive Syllogism (15) Commutativity, Conjunctive Simplification (16) Disjunctive Syllogism

C: (3) Conjunctive Simplification (4) Disjunctive Amplification, Commutativity (5) Conjunction (6) Conditional Proof (7) Commutativity, Conjunctive Simplification (8) Disjunctive Syllogism

D:

(1)

①	$a \rightarrow b$	Premise
②	$b \lor \neg c$	Premise
③	$\neg c \lor b$	Commut.: 2
④	$c \rightarrow b$	Implication: 3
⑤	$(a \lor c) \rightarrow b$	Prf. Cases: 1, 4

(2)

①	$d \land e$	Premise
②	e	Comm., Simpl. 1
③	$e \rightarrow (f \lor g)$	Premise
④	$f \lor g$	Modus Ponens: 2, 3
⑤	$\neg g$	Premise
⑥	f	Comm, Disj. Syll.: 4, 5
⑦	$f \lor h$	Disj. Amp'n.: 6

(3)

①	$k \rightarrow m$	Premise
②	$\neg k \lor m$	Implication: 1
③	$\neg\neg m \lor \neg k$	Comm., Dbl Neg.: 2
④	$\neg m \rightarrow \neg k$	Implication: 3

(4)

①	$n \land (n \lor \neg n)$	Premise
②	$n \land T_0$	Inverse: 1
③	n	Identity
④	$p \rightarrow \neg n$	Premise
⑤	$\neg\neg n$	Dbl.Neg.: 3
⑥	$\neg p$	Modus Tollens: 4, 5

(5)

①	$(q \lor r) \leftrightarrow s$	Premise
②	$[\neg(q \lor r) \lor s] \land [\neg s \lor (q \lor r)]$	Biconditional: 1
③	$\neg(q \lor r) \lor s$	Conj.Simp.: 2
④	$(q \lor r) \rightarrow s$	Implication: 3
⑤	$q \oplus r$	Premise
⑥	$(q \lor r) \land \neg(q \land r)$	Exclusive Or: 5
⑦	$q \lor r$	Conj.Simp.: 6
⑧	s	Modus Ponens: 7, 4

(6)

①	$(t \lor u) \rightarrow (t \lor \neg u)$	Premise
②	u	Premise
③	$t \lor u$	Disj.Amp'n, Comm.: 2
④	$t \lor \neg u$	Modus Ponens: 3, 1
⑤	$(t \lor u) \land (t \lor \neg u)$	Conjunction: 3, 4
⑥	$t \lor (u \land \neg u)$	Distributivity: 5
⑦	$t \lor F_0$	Inverse: 6
⑧	t	Identity: 7

(7)

①	$\neg y$	Premise
②	$y \lor z$	Premise
③	z	Disj.Syll.: 2, 1
④	$z \rightarrow (w \lor x)$	Premise
⑤	$w \lor x$	Modus Ponens: 3, 4
⑥	$\neg w \lor \neg x \lor y$	Premise
⑦	$y \lor (\neg w \lor \neg x)$	Assoc., Comm.: 6
⑧	$\neg w \lor \neg x$	Disj. Syll.: 7, 1
⑨	$\neg(w \land x)$	DeMorgan's: 8
⑩	$(w \lor x) \land \neg(w \land x)$	Conjunction: 5, 9
⑪	$w \oplus x$	Exclusive Or: 10

(8)

①	$a \lor (b \land c)$	Premise
②	$(a \lor b) \land (a \lor c)$	Distributivity: 1
③	$c \lor a$	Comm., Conj.Simp.: 2
④	$\neg\neg c \lor a$	Double Negation: 3
⑤	$\neg c \rightarrow a$	Implication: 4
⑥	$\neg d \rightarrow b$	Premise
⑦	$\neg b \rightarrow \neg\neg d$	*Contrapositive (D(3))*: 6
⑧	$\neg b \rightarrow d$	Double Negation: 7
⑨	$\neg(d \land a)$	Premise
⑩	$\neg d \lor \neg a$	DeMorgan's: 9
⑪	$\neg b \lor \neg c$	Destr.Dilemma: 8, 5, 10

(9)

①	$\neg e \rightarrow (e \lor f)$	Premise
②	$\neg\neg e \lor (e \lor f)$	Implication: 1
③	$(e \lor e) \lor f$	Assoc., Dbl.Neg.: 2
④	$e \lor f$	Idempotent: 3
⑤	$f \lor e$	Commutativity: 4
⑥	$e \rightarrow (f \land g)$	Premise
⑦	$\neg e \lor (f \land g)$	Implication: 6
⑧	$(\neg e \lor f) \land (\neg e \lor g)$	Distributivity: 7
⑨	$\neg e \lor f$	Conj.Simp.: 8
⑩	$f \lor \neg e$	Commutativity: 9
⑪	$(f \lor e) \land (f \lor \neg e)$	Conjunction: 5, 10
⑫	$f \lor (e \land \neg e)$	Distributivity: 11
⑬	$f \lor F_0$	Inverse: 12
⑭	f	Identity: 13