Product Rule & Quotient Rule

PRODUCT RULE

The Product Rule is used to take the derivative of a "product", or in other words the derivative of two functions multiplied together. The general form looks like

$$(f(x)g(x))' = f'(x) \cdot g(x) + g'(x) \cdot f(x)$$

In other words, take the derivative of the first function multiplied by the second function and add it to the derivative of the second function multiplied by the first function. You can make this a bit easier for yourself by writing out a simple table and then plugging those expressions into the equation above.

$f(x) =$	$g(x) =$
$f'(x) =$	$g'(x) =$

Exercise 1: Find the derivative of $F(x) = x^2 \sin x$.

Solution: This function is a product of x^2 and $\sin x$. Let $f(x) = x^2$ and $g(x) = \sin x$. Make the table:

$f(x) = x^2$	$g(x) = \sin x$
$f'(x) = 2x$	$g'(x) = \cos x$

Now plug into the formula:

$$F'(x) = 2x \cdot \sin x + \cos x \cdot x^2$$

The Product Rule is flexible enough that you can do the two halves of the derivative in either order, but if you start with "derivative of the first function" first, it will help you remember the next rule, which is less flexible.

QUOTIENT RULE

The Quotient Rule is used to take the derivative of a "quotient", two functions being divided. The general form looks like:

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - g'(x)f(x)}{[g(x)]^2}$$

You can make the same list as above and then plug into the formula to find the derivative.

Example 2: Find the derivative of $F(x) = \frac{x^2}{\sin x}$

Solution: This function is the quotient of x^2 divided by $\sin x$. Let $f(x) = x^2$ and $g(x) = \sin x$. Make the table:

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$$f(x) = x^2 \qquad g(x) = \sin x$$
$$f'(x) = 2x \qquad g'(x) = \cos x$$

Now plug into the formula:

$$F'(x) = \frac{2x \cdot \sin x - x^2 \cdot \cos x}{(\sin x)^2}$$

We could simplify this further, but at this stage it's more important that you implement these rules correctly. You will not be expected to simplify a result like this on an assignment or test.

EXERCISES

A. If $\frac{d}{dx} \sin x = \cos x$ and if $\frac{d}{dx} \cos x = -\sin x$, use the Product Rule and Quotient Rule to determine the following derivatives:

1) $\frac{d}{dx} \sin 2x = \frac{d}{dx} (2 \sin x \cos x)$
2) $\frac{d}{dx} \cos 2x = \frac{d}{dx} (\cos^2 x - \sin^2 x)$
3) $\frac{d}{dx} \sec x = \frac{d}{dx} \frac{1}{\cos x}$
4) $\frac{d}{dx} \csc x = \frac{d}{dx} \frac{1}{\sin x}$
5) $\frac{d}{dx} \tan x$
6) $\frac{d}{dx} \cot x$

B. Find the derivative of the following functions using the Product Rule. You will need your answers to part A for some problems. Do not simplify your answers.

1) $y = x^3 \tan x$
2) $h(x) = x \ln x$
3) $F(x) = x e^x$
4) $y = e^x \sin x$
5) $h(x) = 3^x \cot x$
6) $y = 4x^{3/2} \sec x$
7) $y = (5 - \sqrt{x})(e^x)$
8) $y = 7v^3 - \sin v \cos v$
9) $G(y) = (y^3 + 4y - 3)(y^2 + 2)$

C. Find the derivative of the following functions using the Quotient Rule. You will need your answers to part A for some problems. Do not simplify your answers.

1) $F(x) = \frac{x}{e^x}$
2) $q(x) = \frac{x^{14}}{x^{13}}$
3) $y = \frac{x + 7}{x^4}$
4) $s(t) = \frac{t^2 - t}{\cos t}$
5) $h(x) = \frac{3^x}{\tan x}$
6) $y = \frac{3 \sin x}{2} + \frac{2 \tan x}{x}$
7) $y = \frac{\sin x + \cos x}{\ln x}$
8) $R(n) = \frac{\cos 2n}{n^2 + (2/n)}$
9) $y = \frac{(0.05 + x)(0.07 + x)}{x^2}$

D. Prove that the following are equal.

1) your answer to B9 and the derivative of $(y^3 + 4y - 3)(y^2 + 2)$ after it's expanded
2) your answer to C2 and the derivative of x^{14} / x^{13} after it's reduced
3) your answer to C3 and the derivative of $(x + 7)/x^4$ after it's divided

4) your answers to B5 and C5

SOLUTIONS

A: (1) $2\cos x \cdot \cos x + 2\sin x(-\sin x) = 2\cos^2 x - 2\sin^2 x$

(2) $(-\sin x)(\cos x) + (\cos x)(-\sin x) - [\cos x \sin x + \sin x \cos x] = -4\sin x \cos x$

(3) $\frac{0 \cdot \cos x - 1 \cdot (-\sin x)}{(\cos x)^2} = \sec x \tan x$ (4) $\frac{0 \cdot \sin x - 1 \cdot \cos x}{(\sin x)^2} = -\cot x \csc x$

(5) $\frac{\cos x \cdot \cos x - \sin x \cdot (-\sin x)}{(\cos x)^2} = 1 + \tan^2 x = \sec^2 x$

(6) $\frac{-\sin x \cdot \sin x - \cos x \cdot \cos x}{(\sin x)^2} = -1 - \cot^2 x = -\csc^2 x$

B: Simplifications are provided for your reference: (1) $y' = 3x^2 \tan x + x^3 \sec^2 x$

(2) $h'(x) = 1 \cdot \ln x + x \cdot \frac{1}{x} = \ln x + 1$ (3) $F'(x) = 1 \cdot e^x + x\, e^x = (x + 1)e^x$

(4) $y' = e^x \sin x + e^x \cos x$ (5) $h'(x) = 3^x \ln 3 \cot x + 3^x (-\csc^2 x)$

(6) $y' = 4(\frac{3}{2} x^{1/2}) \sec x + 4x^{3/2} \sec x \tan x = 6\sqrt{x} \sec x + 4x^{3/2} \sec x \tan x$

(7) $y' = (-\frac{1}{2}x^{-\frac{1}{2}})(e^x) + (e^x)(5 - \sqrt{x})$

(8) $y' = 21v^2 - [(\cos v)(\cos v) + (\sin v)(-\sin v)] = 21v^2 - \cos^2 v + \sin^2 v$

(9) $G'(y) = (3y^2 + 4)(y^2 + 2) + (y^3 + 4y - 3)(2y)$

C: (1) $F'(x) = \frac{1 \cdot e^x - x \cdot e^x}{(e^x)^2} = \frac{1 - x}{e^x}$ (2) $q'(x) = \frac{14x^{13} \cdot x^{13} - x^{14} \cdot 13x^{12}}{(x^{13})^2}$

(3) $y' = \frac{1 \cdot x^4 - (x + 7) \cdot 4x^3}{(x^4)^2}$

(4) $s'(t) = \frac{(2t - 1) \cdot \cos t - (-\sin t) \cdot (t^2 - t)}{(\cos t)^2} = (2t - 1)(\sec t) + (t^2 - 2)(\sec t \tan t)$

(5) $h'(x) = \frac{3^x \ln 3 \tan x - \sec^2 x \cdot 3^x}{(\tan x)^2}$ (6) $y' = \frac{3}{2}\cos x + \frac{2\sec^2 x \cdot x - 2\tan x}{x^2}$

$= \frac{3}{2}\cos x + 2x^{-1}\sec^2 x - 2x^{-2}\tan x$ (7) $y' = \frac{(\cos x - \sin x)(\ln x) - (\sin x + \cos x)(\frac{1}{x})}{(\ln x)^2}$

(8) $R'(n) = \frac{-4\sin n \cos n\, (n^2 + \frac{2}{n}) - \cos 2n\, (2n - \frac{2}{n^2})}{(n^2 + \frac{2}{n})^2}$

(9) $y' = \frac{(2x + 0.12)(x^2) - (2x)(0.05 + x)(0.07 + x)}{(x^2)^2} = -0.12x^{-2} - 0.007x^{-3}$

D: (1) $y^5 + 6y^3 - 3y^2 + 8y - 6$ (2) $\frac{d}{dx}\, x = 1$ (3) $\frac{d}{dx}\,(x^{-3} + 7x^{-4}) = -3x^{-4} - 28x^{-5}$

(4) $3^x \ln 3 \cot x - 3^x \csc^2 x$