Maxima & Minima

Aside from finding the tangent line to a curve, derivatives can also be used to find the highest or lowest value for a function given a set of constraints. For a curve that is continuous over the entire domain of a function, these extreme points, or **extrema** (pl. of extremum), can only occur where the derivative is equal to zero. The **critical values** for a function are those x-values where the derivative of the function is zero, or the derivative does not exist.

If an extreme point is the highest point in its neighbourhood, it's called a **maximum**, and if it's the lowest, it's a **minimum**. If there are other values for $f(x)$ elsewhere that surpass these points, then these are called a **local maximum** or a **local minimum**. If a value for $f(x)$ is the highest or lowest possible for the function, then it is a **global maximum** or a **global minimum**.

Example 1: Find all extreme points on the curve $y = x^4 - 4x^3 - 20x^2 + 5$.

Solution: First, we take the derivative, then we find its zeroes:

$$\begin{aligned} y &= x^4 - 4x^3 - 20x^2 + 5 \\ y' &= 4x^3 - 12x^2 - 40x \\ &= 4x(x^2 - 3x - 10) \\ &= 4x(x-5)(x+2) \\ 0 &\overset{!}{=} 4x(x-5)(x+2) \\ x &= -2, 0 \text{ or } 5 \end{aligned}$$

We plug these values for x into our original equation for y:

$$\begin{aligned} y &= [-2]^4 - 4[-2]^3 - 20[-2]^2 + 5 \\ &= -27 \\ y &= [0]^4 - 4[0]^3 - 20[0]^2 + 5 \\ &= 5 \\ y &= [5]^4 - 4[5]^3 - 20[5]^2 + 5 \\ &= -370 \end{aligned}$$

The critical points are (−2, −27), (0, 5) and (5, −370). We can tell whether these are maxima or minima by applying the Second Derivative Test. If the result is <u>positive</u>, then the point is a <u>minimum</u>. If the result is <u>negative</u>, the point is a <u>maximum</u>. (We'll look at the possibility that the result is zero on the next page.)

$$\begin{aligned} y' &= 4x^3 - 12x^2 - 40x \\ y'' &= 12x^2 - 24x - 40 \end{aligned}$$

$$\begin{aligned} \text{when } x &= -2 \\ y'' &= 12[-2]^2 - 24[-2] - 40 \\ &= 56 \quad \text{minimum!} \end{aligned}$$

$$\begin{aligned} \text{when } x &= 0 \\ y'' &= 12[0]^2 - 24[0] - 40 \\ &= -40 \quad \text{maximum!} \end{aligned}$$

$$\begin{aligned} \text{when } x &= 5 \\ y'' &= 12[5]^2 - 24[5] - 40 \\ &= 140 \quad \text{minimum!} \end{aligned}$$

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Not every point where the derivative is zero is an extreme point, however, and not every extremum occurs where the derivative is zero….

Example 2: Find the global maximum and minimum for the function $y = x^4 - 2x^3 - 36x^2 + 162x + 1$ over the interval [−7, 5].

Solution: Again, we start with the zeroes of the derivative:

$$\begin{aligned} y &= x^4 - 2x^3 - 36x^2 + 162x + 1 \\ y' &= 4x^3 - 6x^2 - 72x + 162 \\ 4x^3 - 6x^2 - 72x + 162 &= 0 \\ 2x^3 - 3x^2 - 36x + 81 &= 0 \end{aligned}$$

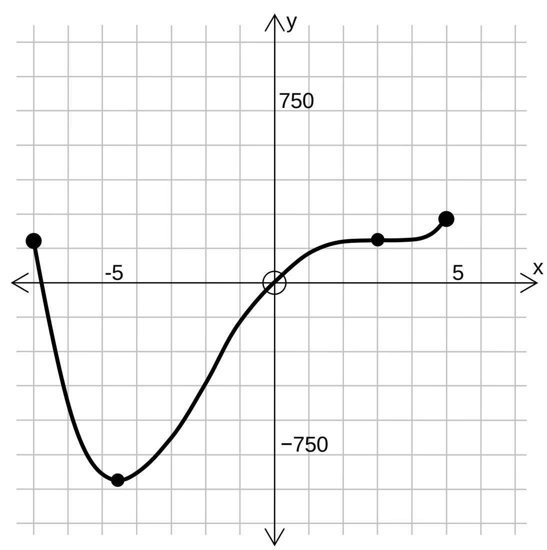
We can use synthetic division to find one factor of this cubic. When we factor, we find that the derivative factors to $2(x - 3)^2(2x + 9)$, so the zeroes are 3 and $-\frac{9}{2}$. These are both within the domain of the function, and they *are* critical points, but have we found a global maximum or minimum?

We take the second derivative and check:

$$y'' = 12x^2 - 12x - 72$$

$$\begin{aligned} 12[3]^2 - 12[3] - 72 &= 0 \quad \text{not a maximum or minimum!} \\ 12[-\tfrac{9}{2}]^2 - 12[-\tfrac{9}{2}] - 72 &= 225 \quad \text{minimum!} \end{aligned}$$

Shown here is the graph of this function. The point at $x = 3$ is called a **saddle point**. The function is increasing on one side of the critical point and decreasing on the other, so it's not a minimum or a maximum. The point at $x = -\frac{9}{2}$ is a minimum, as the test told us.



Recall that the endpoints of a restricted domain (as there is in this question), as well as any x value where there is a discontinuity, are also critical values, and they must be examined. We use the original function to figure out what the global maximum and minimum are, looking at all critical points:

x	y
−7	190
$-\frac{9}{2}$	−858.7
3	190
5	286

So the global minimum is ($-\frac{9}{2}$, −858.6875) and the global maximum is (5, 286).

Note that (5, 286) is *not* a local maximum. An endpoint can never be a local maximum or a local minimum, but it can be a global maximum or minimum. Also note that if the domain had been restricted to [−7, 5), 5 would not be the global maximum, since it wouldn't be within the domain. In a case like this, where the x-value that would otherwise be a global minimum or maximum isn't part of the domain, the function has no global minimum or maximum over that interval.

EXERCISES

A. Find the critical points of the following functions and classify them as maxima, minima or saddle points using the Second Derivative Test:

1) $y = x^2 + 6x - 6$

2) $y = -3x^2 + 9x - 17$

3) $y = 5x^2 + 8x + 13$

4) $y = -\frac{1}{3}x^3 - 2x^2 + 21x + 16$

5) $y = x^3 + 3x^2 + 3x + 3$

6) $y = x^4 + 12x^3 + 48x^2 + 80x + 36$

B. In Math 12, you learned that the vertex of the parabola defined by $y = ax^2 + bx + c$ is at $x = -\frac{b}{2a}$. Use the calculus from this worksheet to prove this fact.

C. Find the global maximum and the global minimum for each function, if possible:

1) $y = x^2 - 2x - 6$, over $[-3, 3]$

2) $y = \dfrac{1}{x^2 + x + 4}$, over $[-2, 2)$

3) $y = |2x| - x - 2$, over $[-1, 2]$

4) $y = -\frac{1}{3}x^3 - 3x^2 - 8x + 16$, over $[-3, 3]$

5) $y = \sin x + x$, over $(\pi, 5\pi]$

SOLUTIONS

A. (1) c.p. = −3 [min] (2) c.p. = $\frac{3}{2}$ [max] (3) c.p. = 10 [min] (4) c.p. = 3 [max], −7 [min] (5) c.p. = −1 [saddle] (6) c.p. = −2 [saddle], −5 [min]

C. (1) g.max = (−3, 9), g.min = (1, −7) (2) g.max = $(-\frac{1}{2}, \frac{4}{15})$, g.min does not exist in this interval (3) g.max = (−1, 1), g.min = (0, −2) (4) g.max = $(-2, \frac{68}{3})$, g.min = (3, −44) (5) g.max = (5π, 5π), g.min does not exist in this interval