Logarithmic Differentiation

Logarithmic differentiation simplifies expressions to make it easier to differentiate them. In both situations when you'll want to use this technique, the steps are the same.

Recall that logarithms are one of three expressions that describe the relationship between three numbers — exponential expressions and radical expressions being the other two. Each of the equations here give the same information, but a different number is isolated in each one. Notice that the isolated number in the logarithmic equation is the exponent in the first. Logarithms are, and behave like, exponents.

$$2^5 = 32 \qquad \sqrt[5]{32} = 2 \qquad \log_2 32 = 5$$

The laws of logarithms reflect this aspect of theirs. The three main ones are listed here, along with the similar laws of exponents. You'll be using the laws to break expressions down, and using the third law here in particular.

$$x^m \cdot x^n = x^{m+n} \qquad \log_B (m \cdot n) = \log_B m + \log_B n$$

$$\frac{x^m}{x^n} = x^{m-n} \qquad \log_B \frac{m}{n} = \log_B m - \log_B n$$

$$(x^m)^n = x^{m \cdot n} \qquad \log_B m^n = n \log_B m$$

DERIVATIVES WHERE LOGARITHMS ARE NECESSARY

By now you should have a healthy catalog of basic functions whose derivatives you know, and rules for manipulating them. Specifically, you know that you can differentiate x^n with the Power Rule (with x being a variable and n being a constant) and n^x as $n^x \ln n$.

Example 1: Differentiate: $(\sin x)^{x^2}$

Solution: We don't have a rule for this kind of derivative, with functions of x in both the base and exponent, but we can make use of the laws of logarithms and implicit differentiation to solve the problem. We'll call that expression whose derivative we're looking for y, and then take the natural logarithm of both sides and apply that third law, so the exponent comes down:

$$\begin{aligned} y &= (\sin x)^{x^2} \\ \ln y &= \ln (\sin x)^{x^2} \\ \ln y &= x^2 \ln (\sin x) \end{aligned}$$

We can now differentiate both sides with respect to x, using implicit differentiation on y, and then isolate y′:

$$\begin{aligned} \tfrac{1}{y}\, y' &= 2x \ln (\sin x) + x^2 \cdot \tfrac{1}{\sin x} \cdot \cos x \\ &= 2x \ln (\sin x) + x^2 \cot x \\ y' &= [2x \ln (\sin x) + x^2 \cot x] \cdot y \\ &= [2x \ln (\sin x) + x^2 \cot x] \cdot (\sin x)^{x^2} \end{aligned}$$

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DERIVATIVES WHERE LOGARITHMS ARE HELPFUL

Example 2: Differentiate: $\frac{(x-5)^7\cos^4 x}{(3x+1)^5}$

Solution: We could differentiate this using Quotient Rule, Product Rule and Power Rule, but the process would be… unpleasant. There would be a lot of moving parts, and there's a good chance we'd miss something and make a mistake. Using logarithmic differentiation here will streamline everything.

The process is the same: call that fraction y, take the natural logarithm of both sides, smash it to pieces with the laws, use implicit differentiation and isolate y′:

$$\begin{aligned} y &= \frac{(x-5)^7\cos^4 x}{(3x+1)^5} \\ \ln y &= \ln \frac{(x-5)^7\cos^4 x}{(3x+1)^5} \\ &= \ln (x-5)^7 + \ln (\cos^4 x) - \ln (3x+1)^5 \\ &= 7 \ln (x-5) + 4 \ln (\cos x) - 5 \ln (3x+1) \\ \frac{1}{y} \cdot y' &= \frac{7}{x-5} + \frac{4}{\cos x} \cdot (-\sin x) - \frac{5}{3x+1} \cdot 3 \\ &= \frac{7}{x-5} - 4 \tan x - \frac{15}{3x+1} \\ y' &= \left(\frac{7}{x-5} - 4 \tan x - \frac{15}{3x+1}\right)\frac{(x-5)^7\cos^4 x}{(3x+1)^5} \end{aligned}$$

We could expand this, and we'd get the version of the answer that matches the result we would get through the regular Rules, but it's not necessary.

EXERCISES

A. Differentiate.

1) $(\tan x)^x$
2) $(x+7)^{x-3}$
3) $[\cos (x+1)]^{\sqrt{x}}$
4) x^{x^2}
5) $(\csc x)^{\csc x}$
6) $(\sin x)^{x \sin x}$

B. Differentiate using logarithmic differentiation.

1) $(x \sin x)^{16}$
2) $\frac{(1+\ln x)^5}{\sec^{12} x}$
3) $(x^3 \tan x)^{10}$
4) $\frac{(2x-5)^7\cos^6 x}{\sin^6 x}$
5) $\sqrt[5]{\sin x \cos^2 x}$
6) $\frac{(3x-1)^8 e^{4x}}{(9x-3)^6}$
7) $\frac{(x^2-3)^9\cot^5 x}{(5-\ln x)^{11}}$
8) $\frac{(2x-3)^8}{(\cos^4 x)(e^x+1)^{13}}$
9) $\sqrt{\frac{(4x+1)^6(\arctan x)^9}{(x^2+1)^8}}$

SOLUTIONS

A: (1) $y' = [\ln (\tan x) + x \cot x](\tan x)^x$ (2) $y' = [\ln (x+7) + \frac{x-3}{x+7}](x+7)^{x-3}$
(3) $y' = \{\frac{\ln [\cos (x+1)]}{2\sqrt{x}} - \sqrt{x} \tan (x+1)\}[\cos (x+1)]^{\sqrt{x}}$ (4) $y' = (2x \ln x + x)(x^{x^2}) = (2 \ln x + 1)(x^{x^2+1})$
(5) $y' = -(\csc x)^{1+\csc x} \cot x [1 + \ln (\csc x)]$ (6) $y' = [(x \cos x + 1) \ln (\sin x) + x \cos x](\sin x)^{x \sin x}$

B: (1) $y' = (16x^{-1} + 16 \cot x)(x \sin x)^{16}$ (2) $y' = (\frac{5}{x + x \ln x} - 12 \tan x) \frac{(1+\ln x)^5}{\sec^{12} x}$
(3) $y' = (30x^{-1} + 10 \cot x \sec^2 x)(x^{30} \tan^{10} x)$ (4) $y' = (\frac{14}{2x-5} - 6 \csc x \cot x)(2x-5)^7 \cot^6 x$
(5) $y' = (\frac{1}{5} \cot x + \frac{2}{5} \tan x)(\sin x \cos^2 x)^{1/5}$ (6) $y' = 3^{-6}(\frac{6}{3x-1} + 4)(3x-1)^2 e^{4x}$
(7) $y' = (\frac{18x}{x^2-3} - 5 \csc x + \frac{11}{5x - x \ln x})\frac{(x^2-3)^9\cot^5 x}{(5-\ln x)^{11}}$ (8) $y' = (\frac{16}{2x-3} + 4 \tan x - \frac{13e^x}{e^x+1})\frac{(2x-3)^8}{(\cos^4 x)(e^x+1)^{13}}$
(9) $y' = (\frac{12}{4x+1} + \frac{9}{2 \arctan x (x^2+1)} - \frac{8x}{x^2+1})\frac{(4x+1)^3(\arctan x)^{9/2}}{(x^2+1)^4}$