# Limits at Infinity

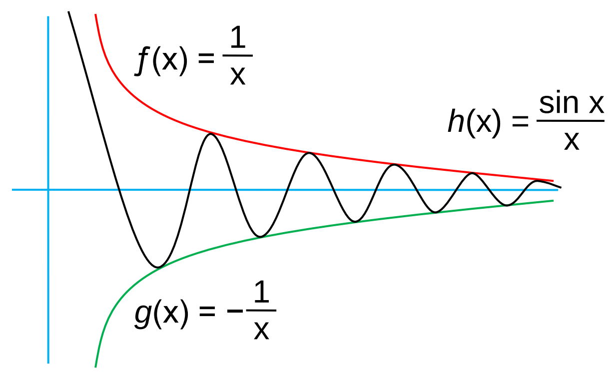
You've seen limit problems like the one at the right where the answer is that it goes to infinity (which also means the limit technically does not exist). This is an **infinite limit**. This worksheet refers to limit problems where x approaches infinity, rather than a finite quantity, like 4, which are called **limits at infinity**.

$$\lim_{x \to 4^+} \frac{x^2 + 10}{x - 4} = \infty$$

If we say $\lim_{x\to k} f(x) = L$, with k being finite, then we mean that when we evaluate $f(x)$ at values that get closer and closer to k, the results get closer and closer to L. So what does $\lim_{x\to\infty} f(x)$ mean? We can't get closer and closer to infinity.

Consider what that limit in the corner is telling us. It says that as we evaluate that rational function at values that get closer and closer to 4 (but always greater than 4), the value of the function increases without bound—the value gets higher and higher. We can use that interpretation to understand the limit at infinity: To say that $\lim_{x\to\infty} f(x) = L$ is to say that as we evaluate $f(x)$ at values of x that are getting higher and higher, the results get closer and closer to L. (A similar definition would explain $\lim_{x\to-\infty} f(x)$.)

Graphically, such a limit *usually* looks like a horizontal asymptote, but it doesn't always. Consider the functions $f(x) = 1/x$, $g(x) = -1/x$, and $h(x) = (\sin x)/x$. They are mapped on the same set of coordinates at the right, and slightly exaggerated. By now, you should know that the curve of $f(x)$ has a horizontal asymptote at $y = 0$, and therefore so should $g(x)$. The curve of $h(x)$ does not have a horizontal asymptote at $y = 0$ — the fact that the curve keeps crossing that line disqualifies it — but is it true that $\lim_{x\to\infty} h(x) = 0$?




Logically, it should be. We can say that the value of $f(x)$ and $g(x)$ get closer to 0 as x increases — it's worth memorizing that $\lim_{x\to\infty} 1/x = 0$ — and these two curves act as upper and lower bounds for $h(x)$. After all, with $x > 0$:

$$-1 \le \sin x \le 1$$
$$-\frac{1}{x} \le \frac{\sin x}{x} \le \frac{1}{x}$$

By the same logic that gave us the Squeeze Theorem, $h(x)$ should also get closer to 0 *on average* as x increases. We will want the exact definition of a limit at infinity to include this idea.

Limits at infinity are inherently one-sided limits. We can only think of approaching a limit as $x \to \infty$ from the left, and as $x \to -\infty$ from the right. As such, there's no requirement that the two one-sided limits must agree; the only way a limit at infinity doesn't exist is if it does not settle on a finite result. Such a limit will either increase without bound or



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decrease without bound.

**THE PRECISE DEFINITION**

There is no epsilon-delta definition for a limit at infinity, but it has a similar philosophy. It still gives a maximum tolerance, ε, for the limit (how far from L am I allowed to be?) and requires that we provide a point past which that tolerance will always be met. For the epsilon-delta version, we say that so long as x is no more than δ away from k, then $f(x)$ will be no more than ε away from L. For limits at infinity, we give a minimum value M, and as long as x is greater than M, then $f(x)$ will be no more than ε away from L.

*Example 1:* Prove that $\lim_{x\to\infty} (\sin x)/x = 0$ using the formal definition of a limit.

*Solution:* Imagine we had a heckler, someone who didn't believe us about the limit being zero. The heckler says, "I won't believe you about the limit being zero unless you can get within 0.01 of zero and stay there!" We can meet his challenge. We know that our curve is bounded by $y = 1/x$ and $y = -1/x$. Since $1/x$ is a decreasing and positive function over the positive real numbers, when $x > 100$, $0.01 > 1/x > 0$. For any value past $x = 100$, $1/x$ is within our heckler's stated tolerance. A similar argument can be made for $y = -1/x$. We've shown that $y = (\sin x)/x$ is no further away from zero than these two curves are, so $y = (\sin x)/x$ must also be within tolerance.

Our response to the heckler, then, is, "We can get y within 0.01 of zero, if you set x to a value where x > 100."

The heckler shouts back, "Oh yeah? Well, I won't believe you unless you can get within 0.00001 of zero and stay there!"

We say, "That was, what, four zeroes after the decimal? One over ten thousand? Fine. As long as you set x > 10,000, we're within tolerance again."

"Oh *yeah?* Well—"

"Look. No matter what number you name, we can be within tolerance by taking your number, finding its reciprocal, and making that our new minimum value. Since this is supposed to be a *formal* proof, if you name a positive number ε, no matter how small, and you require $|f(x) - 0| < \varepsilon$ for all $x > M$, then we can calculate the value of M as $M = 1/\varepsilon$. Then:

$$x > M \Rightarrow x > 1/\varepsilon \Rightarrow 1/x < \varepsilon, \text{ since x is positive}$$

$$\therefore -\varepsilon < -\frac{1}{x} \leq \frac{\sin x}{x} \leq \frac{1}{x} < \varepsilon$$

$$\therefore \left|\frac{\sin x}{x}\right| < \varepsilon \Rightarrow \left|\frac{\sin x}{x} - 0\right| < \varepsilon \Rightarrow |f(x) - L| < \varepsilon$$

…and the requirements of the definition of a limit at infinity are met. Also, moving the goalposts is a logical fallacy, so stop arguing. We can beat you no matter what you say." The heckler is defeated since we've shown he can always have his demands met by writing a generic expression for M in terms of ε that always leads back to the statement of tolerance, $|f(x) - L| < \varepsilon$. The limit really does equal zero.

This formal definition of a limit is also the definition of an asymptote, but without the absolute value signs. That's why horizontal asymptotes can be "crossed" at low, finite values of x — If $f(c) = L$, but $c < M$, it doesn't interfere with the definition.

*Example 2:* Find $\lim_{x \to \infty} \frac{3x^3 - 5x + 9}{2x^3 - 4x^2 + 7}$.

*Solution:* You looked at the end behaviour of rational expressions like this in Grade 12 Math. Recall:

- If the degree of the numerator is greater than the degree of the denominator, then the curve goes to infinity (or negative infinity).
- If the degree of the numerator is *one more* than the degree of the denominator, then the curve has a slant asymptote, found by performing polynomial division.
- If the degree of the denominator is greater than the degree of the numerator, then the curve has a horizontal asymptote at $y = 0$.
- If the degrees of the numerator and denominator are equal, the curve has a horizontal asymptote at $y = k$, where k is the ratio of the leading coefficients in the expression.

The question here falls into the last category; the answer should be $\frac{3}{2}$. We need to be able to show this with a limit, however. Here's the more rigorous method: multiply the fraction by $x^{-b}/x^{-b}$ where b is the degree of the denominator. We get:

$$\lim_{x \to \infty} \frac{3x^3 - 5x + 9}{2x^3 - 4x^2 + 7} = \lim_{x \to \infty} \frac{3 \cdot x^3/x^3 - 5 \cdot x/x^3 + 9 \cdot 1/x^3}{2 \cdot x^3/x^3 - 4 \cdot x^2/x^3 + 7 \cdot 1/x^3} = \lim_{x \to \infty} \frac{3 - 5 \cdot 1/x^2 + 9 \cdot 1/x^3}{2 - 4 \cdot 1/x + 7 \cdot 1/x^3}$$

$$= \frac{3 - 0 + 0}{2 - 0 + 0} = \frac{3}{2}$$

We can justify the step where the limit is evaluated by using the Limit Laws and breaking it down.

Why do rational functions behave this way? The degree of the polynomial is an indication of how fast it increases toward infinity, and the trailing terms (all terms other than the leading term) don't contribute to that in any meaningful way.

If the numerator is of higher degree than the denominator, the denominator quickly shrinks to insignificance just like the trailing terms do, and the function simply increases (or if negative, decreases).

If the denominator has the higher degree, the same thing happens, with a small number of a very large number, which has the same result as $\lim_{x\to\infty} 1/x$: it tends to 0.

Something more interesting happens when the degrees are equal. Both the numerator and denominator increase at more or less the same rate, with one tending to be a *finite* multiple of the other. The higher the value of x, the closer this statement gets to being true (as the trailing terms become less important in evaluating the fraction), and so the limit at infinity takes on the value of that finite multiplier.

## LIMITS WORTH KNOWING

$\lim_{x\to 0} 1/x = \infty$ $\lim_{x\to\infty} 1/x = 0$ $\lim_{x\to\pm\infty} \arctan x = \pm\pi/2$ $\lim_{x\to-\infty} e^x = 0$

## EXERCISES

A. 1) Without using calculus, what is the location of the horizontal asymptotes of the graph of $y = \dfrac{3x^3 + 6x^2 + 14x + 84}{(x + 7)(x - 2)^2}$?

2) Find the y-intercept of the graph in part (1).

3) Does your answer to (2) mean that the limit at infinity of y should not be the same as your answer to (1)? Why or why not?

4) If you graph this curve at desmos.com, you'll see that for values of x greater than 2, the curve is strictly decreasing. Use this graph to find a minimum value for M which can be used to prove your answer to $\lim_{x\to\infty} y$ for $\varepsilon = 0.09$.

B. Your classmate Jason looks at the question, "Find $\lim_{x\to\infty} \dfrac{9x^4 - 6x^3 + 13x^2 - 3x + 7}{3x^3 - x^2 + 7x + 2}$."
He divides the numerator by the denominator and gets the result $3x + 1$ plus a remainder. He says the answer to the limit is therefore $3x + 1$. Your other classmate Sonja says that the limit simply doesn't exist. Who (if anyone) is right?

C. Graph $y = x^2$ and $y = \ln x$ at desmos.com. Use this graph to find $\lim_{x\to\infty} \dfrac{\ln x}{x^2}$.

D. Your classmate Janoš says that $\lim_{x\to\infty} (\sin x) = 0$, since it oscillates around this value. Use the precise definition of a limit to prove that the limit cannot equal zero.

E. If $f(x) = P(x)/Q(x)$, where $P(x)$ and $Q(x)$ are polynomials, and $\lim_{x\to\infty} f(x) = L$, $L \in \mathbb{R}$, is it possible that $\lim_{x\to-\infty} f(x) \neq L$? Why or why not?

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## SOLUTIONS

A: (1) $y = 3$, since the numerator and denominator are both degree 3 (2) (0, 3)
(3) The limit only looks at large values of x, so the value at $x = 0$ isn't relevant. Any value for M in the formal definition of the limit would have to be greater than 2, the location of the vertical asymptote. (4) You can use $M = 18$, since the curve passes through (18, 3.09) exactly.

B: Sonja is right. The degree of the numerator is higher than the denominator, so the answer is $\infty$. Jason has found the equation of the slant asymptote, and the curve approximates that line for large values of x, but $\lim_{x\to\infty} (3x + 1)$ is $\infty$. The limit tends to infinity for both expressions, and so neither limit exists. The answer to a limit question is either a number or "D.N.E.", never an expression in x.

C: Since the denominator increases much faster than the numerator, the limit is 0.

D: Consider the case where $\varepsilon = 1$. Then we need a value M such that $\sin x < 1$ for all $x > M$, but there are infinitely many solutions to $\sin x = 1$, in the form $x = 2k\pi$, $k \in \mathbb{Z}$. For any positive value of $\varepsilon$ that is less than 1, there will also be infinitely many solutions to $\sin x = \varepsilon$. Since $\varepsilon$ cannot be made arbitrarily small, the requirements of the precise definition of a limit cannot be met.

E: It's not possible. If the degree of $P(x)$ is higher than $Q(x)$, the limit wouldn't be a real number. If $Q(x)$ is higher degree than $P(x)$, both sides will tend to zero. If the degrees are equal, the variable parts of the leading term will effectively cancel; the change in the sign of x won't change the ratio of the leading coefficients.