



Limit Problems

The limit is a basic calculus concept that shows up again and again. Remember that a limit only looks at values of a function *near* the one written under the limit sign.

PLUG-IN PROBLEMS

Example 1: Evaluate: $\lim_{x \rightarrow 4} (x^2 + 5)$

Solution: The function $x^2 + 5$ is continuous, so plugging 4 in for x produces a real answer, and that's the value of the limit:

$$\lim_{x \rightarrow 4} (x^2 + 5) = [4]^2 + 5 = 21$$

INDETERMINATE LIMITS

Example 2: Evaluate: $\lim_{x \rightarrow 2} \frac{x^2 + 3x - 10}{x - 2}$

Solution: Plugging in 2 for x would result in $0/0$. Whenever we see this value, it means that there must be a factor that can cancel out so we can plug in for x and get a real answer. There must be a factor of $(x - 2)$ within the fraction and we should factor so that we can cancel it:

$$\begin{aligned} \text{when } x \neq 2, \frac{x^2 + 3x - 10}{x - 2} &= \frac{(x + 5)(x - 2)}{x - 2} \\ &= x + 5 \\ \therefore \lim_{x \rightarrow 2} \frac{x^2 + 3x - 10}{x - 2} &= \lim_{x \rightarrow 2} (x + 5) \\ &= [2] + 5 = 7 \end{aligned}$$

Example 3: Evaluate: $\lim_{x \rightarrow 4} \frac{\sqrt{x + 12} - x}{x - 4}$

Solution: Plugging in 4 for x again results in $0/0$. It's not obvious that we can cancel $(x - 4)$, but it is still possible. In a situation like this, we multiply the fraction by the conjugate of the numerator:

$$\begin{aligned} \text{when } x \neq 4, \frac{\sqrt{x + 12} - x}{x - 4} &= \frac{\sqrt{x + 12} - x}{x - 4} \cdot \frac{\sqrt{x + 12} + x}{\sqrt{x + 12} + x} \\ &= \frac{x + 12 - x^2}{(x - 4)(\sqrt{x + 12} + x)} \\ &= \frac{-(x - 4)(x + 3)}{(x - 4)(\sqrt{x + 12} + x)} \\ &= \frac{-x - 3}{\sqrt{x + 12} + x} \\ \therefore \lim_{x \rightarrow 4} \frac{\sqrt{x + 12} - x}{x - 4} &= \lim_{x \rightarrow 4} \frac{-x - 3}{\sqrt{x + 12} + x} \\ &= \frac{-[4] - 3}{\sqrt{[4] + 12} + [4]} = -\frac{7}{8} \end{aligned}$$



DIVIDING BY ZERO: VERTICAL ASYMPTOTES

Example 4: Evaluate: $\lim_{x \rightarrow 3} \frac{x-5}{x-3}$

Solution: If we plug 3 into this problem, we get $\frac{-2}{0}$. We can't cancel anything to fix the problem here. An answer in the form $\frac{\neq 0}{0}$ suggests that there is a vertical asymptote. For this problem, $x = 3$ is the location of a vertical asymptote.

We branch the problem off into its left-hand and right-hand limits. Rather than using specific values higher or lower than 3, we'll consider only the trend of the calculations for values near 3. We'll use the convention that 3^+ is a number *very slightly* more than 3, and 3^- is a number *very slightly* less than 3. Any calculation involving one of these numbers will give the answer we expect (so that $7 - 2^+ \approx 5$). Once we have that result we need to consider whether the answer is slightly more than or slightly less than the result (so that $7 - 2^+ = 5^-$). In a part of a function that doesn't affect its limit, we can forgo these approximations (as we will in the numerator of this example):

$$\lim_{x \rightarrow 3^+} \frac{x-5}{x-3} \rightarrow \frac{3-5}{3^+-3} = \frac{-2}{0^+}$$

$$\lim_{x \rightarrow 3^-} \frac{x-5}{x-3} \rightarrow \frac{3-5}{3^- - 3} = \frac{-2}{0^-}$$

We can now finish using $\lim_{x \rightarrow 0} \frac{1}{x} = \infty$ and $\lim_{x \rightarrow \infty} \frac{1}{x} = 0$. The right-hand limit is a negative number over a positive, small number: $\lim_{x \rightarrow 3^+} \frac{x-5}{x-3} = -\infty$. The left-hand limit is a negative number over a negative, small number: $\lim_{x \rightarrow 3^-} \frac{x-5}{x-3} = \infty$. The left-side and right-side limits don't agree, so the limit does not exist.

Note that 3^- is very different from -3 ! The value of " 3^- " is a *positive* number close to 3. In all cases other than 0, the approximations have the same sign as the number they're approximating. For 0^+ and 0^- , the approximations have the same sign as the superscript.

We can also use this branching approach in piecewise functions, absolute value expressions and other situations where the overall limit cannot be evaluated directly.

Example 5: Evaluate: $\lim_{x \rightarrow -4} \frac{x+5}{x^3 + 6x^2 - 32}$

Solution: If we plug -4 into this problem, we get $\frac{1}{0}$. We'll use the same branching and trending technique as the previous problem, but it's difficult to compare -4^+ between the two terms containing an x in the denominator. However we do know that $(x + 4)$ must be a factor in the denominator. If we factor completely, we can use the trending technique:

$$\frac{x+5}{x^3 + 6x^2 - 32} = \frac{x+5}{(x+4)^2(x-2)}$$

$$\lim_{x \rightarrow -4^+} \frac{x+5}{(x+4)^2(x-2)} \rightarrow \frac{1}{(-4^+ + 4)^2(-6)} = \frac{1}{(0^+)^2(-6)} = \frac{1}{(0^+)(-6)} = \frac{1}{0^-} = -\infty$$

$$\lim_{x \rightarrow -4^-} \frac{x+5}{(x+4)^2(x-2)} \rightarrow \frac{1}{(-4^- + 4)^2(-6)} = \frac{1}{(0^-)^2(-6)} = \frac{1}{(0^-)(-6)} = \frac{1}{0^+} = -\infty$$



The two one-sided limits agree, so $\lim_{x \rightarrow -4} \frac{x+5}{x^3+6x^2-32} = -\infty$

APPROACHING INFINITY: HORIZONTAL LIMITS

Example 6: Evaluate: $\lim_{x \rightarrow \infty} \frac{3x^2 + x - 17}{x^2 - 5x + 13}$

Solution: We can use limit laws to rewrite this limit, and the fact that $\lim_{x \rightarrow \infty} \frac{1}{x} = 0$ to find the value of this limit. We divide the numerator and the denominator by the highest order power of x from the denominator. In this case, that's x^2 . (Don't include the coefficient when you do this. It won't be wrong, but it'll make things harder.)

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{3x^2 + x - 17}{x^2 - 5x + 13} &= \lim_{x \rightarrow \infty} \frac{3\frac{x^2}{x^2} + \frac{x}{x^2} - 17\frac{1}{x^2}}{\frac{x^2}{x^2} - 5\frac{x}{x^2} + 13\frac{1}{x^2}} \\ &= \lim_{x \rightarrow \infty} \frac{3 + \frac{1}{x} - 17\frac{1}{x^2}}{1 - 5\frac{1}{x} + 13\frac{1}{x^2}} = \frac{3 + 0 - 0}{1 - 0 + 0} = 3 \end{aligned}$$

When you're examining the limit of a rational function of polynomials as $x \rightarrow \infty$ you can figure out what answer you should get by looking at the leading terms of the polynomials.

- If the numerator has a higher power than the denominator, then the function simply increases, and the limit is ∞ .
- If the numerator has a lower power than the denominator, then the function has a horizontal asymptote at $y = 0$, and the limit is 0 .
- If the numerator has the same power as the denominator, then the function has a horizontal asymptote elsewhere, and the limit is equal to the ratio of the leading coefficients.

COMPOSITE LIMITS

Example 7: Evaluate: $\lim_{x \rightarrow \infty} \ln\left(\frac{x^3+4}{x^3-4}\right)$

Solution: Here we have a function composed of a logarithmic function and a polynomial rational function. Both of these are in the list of continuous functions in the textbook, and so the following theorem applies:

$$\lim_{x \rightarrow a} f(g(x)) = f(\lim_{x \rightarrow a} g(x))$$

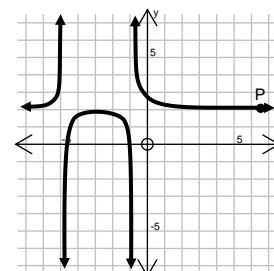
To evaluate a limit of this type, we start with the function on the inside, evaluate its limit and work our way out.

$$\begin{aligned} \lim_{x \rightarrow \infty} \ln\left(\frac{x^3 + 4}{x^3 - 4}\right) &= \ln\left(\lim_{x \rightarrow \infty} \frac{x^3 + 4}{x^3 - 4}\right) \\ &= \ln\left(\lim_{x \rightarrow \infty} \frac{1 + \frac{4}{x^3}}{1 - \frac{4}{x^3}}\right) \\ &= \ln\left(\frac{1+0}{1-0}\right) = \ln 1 = 0 \end{aligned}$$

EXERCISES

A. Shown on the right is the graph of $f(x) = \frac{2x^2 + 12x + 11}{x^2 + 6x + 5}$.

1) A student is asked to evaluate $\lim_{x \rightarrow -5^+} f(x)$. He puts his finger on the point marked P and moves it leftwards along the



curve towards $x = -5$. What's wrong with this procedure?

B. Evaluate the limits:

1) $\lim_{x \rightarrow -1} \frac{4}{x+3}$

3) $\lim_{x \rightarrow 20} \frac{3x^2 - 100}{x^2 + 50}$

5) $\lim_{x \rightarrow 0} e^x$

2) $\lim_{x \rightarrow -1} \frac{x-2}{x^2 + 10x - 24}$

4) $\lim_{x \rightarrow 0} \frac{x^2 + 3x - 28}{x-4}$

6) $\lim_{x \rightarrow 1} \frac{2x^3 - 13x^2 - 22x + 105}{x^3 - 3x^2 - 25x - 21}$

C. Evaluate the limits:

1) $\lim_{x \rightarrow 2} \frac{x-2}{x^2 + 10x - 24}$

4) $\lim_{x \rightarrow -2} \frac{x^3 + 8}{x+2}$

7) $\lim_{x \rightarrow -3} \frac{\sqrt{2x+15} + x}{x+3}$

2) $\lim_{x \rightarrow 4} \frac{x^2 + 3x - 28}{x-4}$

5) $\lim_{x \rightarrow 7} \frac{2x^3 - 13x^2 - 22x + 105}{x^3 - 3x^2 - 25x - 21}$

8) $\lim_{x \rightarrow 3} \frac{x - \sqrt{x+6}}{x^2 + 5x - 24}$

3) $\lim_{x \rightarrow 1} \frac{x^2 + 5x - 6}{2x^2 + x - 3}$

6) $\lim_{x \rightarrow -3} \frac{2x^3 - 13x^2 - 22x + 105}{x^3 - 3x^2 - 25x - 21}$

D. Evaluate the following:

1) $4^+ + 3$

4) $(4^-)^2$

7) $10 - 0^-$

2) $7 - 4^-$

5) $5 + 0^+$

8) $3 \div 0^+$

3) $-8 - 4^+$

6) $-6 + 0^-$

E. Evaluate the limits, if they exist. If they do not exist, state which side of the limit goes to $-\infty$:

1) $\lim_{x \rightarrow -3} \frac{4}{x+3}$

3) $\lim_{x \rightarrow 2} \frac{x+1}{x^3 - 12x + 16}$

2) $\lim_{x \rightarrow 1} \frac{6}{(x-1)^4}$

4) $\lim_{x \rightarrow -4} \frac{x+1}{x^3 - 12x + 16}$

F. Evaluate the limits:

1) $\lim_{x \rightarrow \infty} \frac{3x^2 - 100}{x^2 + 50}$

3) $\lim_{x \rightarrow \infty} \frac{x^2 + 3x - 28}{x-4}$

5) $\lim_{x \rightarrow \infty} \frac{x^3 + 8}{x+2}$

2) $\lim_{x \rightarrow -\infty} \frac{x-2}{x^2 + 10x - 24}$

4) $\lim_{x \rightarrow -\infty} \frac{2x^3 - 13x^2 - 22x + 105}{x^3 - 3x^2 - 25x - 21}$

6) $\lim_{x \rightarrow -\infty} e^x$

G. Evaluate the limits:

1) $\lim_{x \rightarrow 2} \sin(3x^2 - 7)$

3) $\lim_{x \rightarrow 7^+} \ln\left(\frac{x^3 - 19x^2 + 119x - 245}{x-7}\right)$

2) $\lim_{x \rightarrow -6} e^{\frac{x^2 + 13x + 42}{x+6}}$

4) $\lim_{x \rightarrow -\infty} \frac{2}{\pi/2 + \tan^{-1} x}$ [Hint: infinite limits are inherently one-sided]

SOLUTIONS

A. Limits should be evaluated close to the number that x approaches. 6 is not close to -5 , so starting at the right side of the graph doesn't tell you any useful information.

B. (1) 2 (2) $\frac{1}{11}$ (3) $\frac{22}{9}$ (4) 7 (5) 1 (6) $-\frac{3}{2}$

C. (1) $\frac{1}{14}$ (2) 11 (3) $\frac{7}{5}$ (4) 12 (5) $\frac{9}{8}$ (6) $1\frac{1}{2}$ (7) $\frac{4}{3}$ (8) $\frac{5}{66}$

D. (1) 7^+ (2) 3^+ (3) -12^- (4) 16^- (5) 5^+ (6) -6^- (7) 10^+ (8) ∞

E. (1) DNE; $f(x) \rightarrow -\infty$ as $x \rightarrow -3^-$ (2) ∞ (3) ∞ (4) DNE; $f(x) \rightarrow -\infty$ as $x \rightarrow -4^+$

F. (1) 3 (2) 0 (3) ∞ (4) 2 (5) ∞ (6) 0

G. (1) -0.95892 (2) e (3) $-\infty$ (4) ∞

