Implicit Differentiation

Sometimes we need to take the derivative of something that is not a function, but a relation between x and y where it's not possible to isolate y. In these cases we need to extend the Chain Rule to variables other than x. This method is called **implicit differentiation** because we cannot take the derivative of y directly, but the derivative of the expression implies what y′ must be.

Example 1: Find y′: $\sin(x + y) = e^{x-y}$.

Solution: It's very difficult to isolate y in this expression. A simpler solution is to take the derivative of both sides of the equation without isolating y. Since we're performing the same operation on both sides, our new equation will still hold true.

The obvious problem is that we don't know what the derivative of y is. We do know, however, that y is defined in terms of x. We start with the Chain Rule:

Derivative of the left-hand side: $\cos(x + y) \cdot (1 + \ldots?)$
Derivative of the right-hand side: $e^{x-y} \cdot (1 - \ldots?)$

In both cases, the Chain Rule requires us to take the derivative of y. Since y is *not* a constant, we can't write 0. We need whatever the derivative of y is, and we call that y′ (or $\frac{dy}{dx}$, or whatever is most convenient), so that's what we write:

$$\cos(x + y) \cdot (1 + y') = e^{x-y} \cdot (1 - y')$$

Now we can isolate y′ by bringing every term that has y′ to one side, and everything that doesn't to the other:

$$\begin{aligned} \cos(x + y) + y'\cos(x + y) &= e^{x-y} - y' e^{x-y} \\ y'\cos(x + y) + y' e^{x-y} &= e^{x-y} - \cos(x + y) \\ y'[\cos(x + y) + e^{x-y}] &= e^{x-y} - \cos(x + y) \\ y' &= \frac{e^{x-y} - \cos(x+y)}{e^{x-y} + \cos(x+y)} \end{aligned}$$

If we need to take the derivative of an expression in terms of y, we simply use the Chain Rule as usual, and we "chain out" a y′ at the end.

Example 2: Find y″: $\cos x + \sin y = y$

Solution: We'll need to take the first derivative to start with:

$$\begin{aligned} -\sin x + \cos y \cdot y' &= y' \\ -\sin x &= y' - \cos y \cdot y' \\ -\sin x &= (1 - \cos y) \cdot y' \\ y' &= \frac{-\sin x}{1 - \cos y} \end{aligned}$$

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Now we take the derivative again.

$$y'' = \frac{(-\cos x)(1-\cos y)-(\sin y \cdot y')(-\sin x)}{(1-\cos y)^2}$$

$$= \frac{-\cos x+\cos x\cos y+\sin x\sin y \cdot y'}{(1-\cos y)^2}$$

This expression contains a y′, which we can replace with the first derivative to get an answer in terms of just x and y:

$$= \frac{-\cos x+\cos x\cos y}{(1-\cos y)^2}+\frac{\sin x\sin y}{(1-\cos y)^2}\cdot y'$$

$$= \frac{-\cos x-\cos x\cos y}{(1-\cos y)^2}+\frac{\sin x\sin y}{(1-\cos y)^2}\cdot\frac{-\sin x}{1-\cos y}$$

$$= \frac{(-\cos x-\cos x\cos y)(1-\cos y)}{(1-\cos y)^3}-\frac{\sin^2 x\sin y}{(1-\cos y)^3}$$

$$= \frac{-\cos x+\cos x\cos y-\cos x\cos y+\cos x\cos^2 y}{(1-\cos y)^3}-\frac{\sin^2 x\sin y}{(1-\cos y)^3}$$

$$= \frac{-\cos x+\cos x\cos^2 y-\sin^2 x\sin y}{(1-\cos y)^3}$$

EXERCISES

A. Find y′:

1) $\cos(x+y) = 3xy^2$

2) $\dfrac{3x+2}{y} = \ln xy$

3) $\tan y - x\sin y = x^2$

4) $e^{4x-2y} = (x+y)^2$

5) $\dfrac{\sin y}{x^2} = \tan y^4$

6) $\sin x\cos y = \tan xy$

B. Find y″:

1) $y^2 - x^2 = xy$

2) $\cos(x-y) = x^2$

SOLUTIONS

A. (1) $y' = -\dfrac{\sin(x+y)+3y^2}{\sin(x+y)+6xy}$ (2) $y' = \dfrac{3xy-y^2}{3x^2+xy+2x}$ (3) $y' = \dfrac{2x+\sin y}{\sec^2 y-x\cos y}$

(4) $y' = \dfrac{2e^{4x-2y}-x-y}{e^{4x-2y}+x+y}$ (5) $y' = -\dfrac{2\sin y}{4x^3y^3\sec^2 y^4-x\cos y}$ (6) $\dfrac{\cos x\cos y-y\sec^2(xy)}{\sin x\sin y+x\sec^2(xy)}$

B. (1) $y'' = \dfrac{-10x^2-10xy+10y^2}{(2y-3)^3}$ (2) $y'' = \dfrac{4x^2\cos(x-y)+2\sin^2(x-y)}{\sin^3(x-y)}$