Essentials of Calculus 2

Continuity & Differentiability

By now you know that evaluating the limit of a function at a particular value usually just boils down to plugging in that number, at least for most curves that we're likely to encounter. A limit might fail to exist at a vertical asymptote or if there's a "hole" in the graph (both of which can happen with a rational function) or at the value where two pieces of a piecewise function come in contact. (We would also have limits failing to exist if there's an interval outside the domain of the function, such as $\lim_{x\to 0} \sqrt{x^2 - 4}$, but in that case the limit is pretty much meaningless; in the other cases, we might expect the limit to work out, depending on the details going in.)

If a function doesn't have any of these problems, then graphing that function tends to mean that it's one smooth curve that can be graphed without lifting your pencil. We call that a **continuous function**. (If it's not, it's a **discontinuous function**.) The proper definition of a continuous function is that the function is continuous at every point in its domain. A function is continuous at a point (an x-value, really) if three things are true:

- The function can be evaluated at that point, i.e. $f(a)$ exists
- The limit of the function exists at that point, i.e. $\lim_{x\to a} f(x)$ exists
- Those two things agree with each other, i.e. $\lim_{x\to a} f(x) = f(a)$

That third statement is pretty powerful. If a function is known to be continuous, then evaluating a limit is as simple as plugging the number in. The older Stewart textbook had a series of theorems with very useful results, summarized here:

The following types of functions are continuous at every number in their domains:

- polynomials
- rational functions
- root functions
- trigonometric functions
- inverse trigonometric functions
- exponential functions
- logarithmic functions

Also, any sum $(f + g)$, difference $(f - g)$, scalar multiple (cf), or product (fg) of these functions is continuous over their domains. Any quotient of these functions (f/g) is continuous where $g(a) \neq 0$. If g is continuous at a and f is continuous at $g(a)$, then $f(g(a))$ is continuous at a.

We do a number of problems in Calculus 1 which approach proofs, and it's useful to be able to hand-wave and say that we know that certain types of functions are continuous because Stewart says so.

If a function has a "hole" problem at a point, where altering the function by assigning a value (or a new value) to the function for that value, then the original function has a **removable discontinuity**. If the pieces of a piecewise function don't line up, then the function has a **jump discontinuity**.

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The **Intermediate Value Theorem (IVT)** tells us that if we have a continuous function which has two distinct outputs y_1 and y_2 for two distinct inputs, then somewhere between those inputs there must be a point where the function outputs every possible value between y_1 and y_2. It doesn't tell you where; just that the point exists. This doesn't feel like much of a useful theorem — for most functions we've dealt with we could just solve, but it doesn't take much to come up with a function where solving algebraically doesn't work anymore, say, $f(x) = \sin x + \ln x$. Does this function have a solution, a value for x where $\sin x + \ln x = 0$? By the box on the previous page, it's continuous on the interval $(0, \infty)$, which is its domain. (See? Useful!) We can try some numbers. $f(0.001) = -6.9068\ldots$ and $f(1) = +0.84147\ldots$. One result was positive and one was negative. All the conditions of IVT are met, and we have 0 between y_1 and y_2, so we know that a solution exists for x between 0.001 and 1. We don't know where, but we can start closing in to get any degree of precision by tightening that window between 0.001 and 1.

We'll never get the exact answer to that question, but we can approximate, and for many applications, that approximation can be good enough (such as for science or engineering purposes).

DIFFERENTIABILITY

The concept of differentiability is just as easy to understand, but a bit harder to show. If a function is **differentiable**, then you can take its derivative anywhere on its domain. ("Differentiable at a point" has the obvious related definition.) The trouble is that the textbook doesn't give a test for differentiability — if you can take the derivative, it's differentiable.

It's kind of defined by contrast.

- If the function is discontinuous at a point, it cannot be differentiable at that point. (In fact, if you know it *is* differentiable at that point, that is sufficient to prove it's continuous at that point).
- If the function has a point where the curve becomes vertical, the function is not differentiable at that point. The curve $y = x^3$ flattens out at (0, 0) so the slope there is horizontal. Then the inverse function $y = \sqrt[3]{x}$ has a vertical slope at $x = 0$. The curve is not differentiable here.
- If the function has a sharp corner or point, it's not differentiable at that point.

Consider $y = |x|$, the absolute value function. For $x \geq 0$, the graph behaves like $y = x$, and the slope is 1. For $x < 0$, the graph behaves like $y = -x$, and the slope is -1. Because the slopes coming together at $x = 0$ disagree, there's no single slope and the "curve" is not differentiable at $x = 0$.

This may remind us of having two one-sided limits disagree, resulting in an overall limit that does not exist. We can make use of this to create a test for differentiability:

A function $f(x)$ is differentiable at a point only if the associated difference quotient, $[f(x + h) - f(x)]/h$ is continuous at that point.