Systems of Three Equations

Systems of equations with three variables are solved just like systems with two variables. The strategy is the same: perform operations to remove one variable at a time (and the number of equations will go down by one). Eventually, we will get to one equation and one variable, which we know how to solve. We can then go back through all the variables we removed to find their values and solve the system.

ELIMINATION METHOD

[1] **Decide on a variable to eliminate.** Any of the three variables may be chosen, but the right choice can decrease the number of calculations. One way is to look for variables with coefficients of either 1 or −1 and try to eliminate those.

[2] **Add any *two* of the equations to get a new equation with *two* variables.**

[3] **Use a different pair of equations to get a second new equation with the *same* two variables.** If you eliminate a different variable, you're back where you started.

[4] **Solve the resulting pair of new equations** just as you've done before. You'll get the values of two of the variables.

[5] **Substitute into any of the original three equations to find the value of the last variable.**

[6] **Check the solution** to see that it satisfies all three of the original equations. Don't use your two new equations; if you made a mistake when you created them, you won't find it.

Example 1: Solve using the elimination method:

$$\begin{aligned} x + 2y - z &= 0 \quad ① \\ 2x - y + z &= 5 \quad ② \\ 4x + 2y + 5z &= 6 \quad ③ \end{aligned}$$

Solution: [1] Let's choose to eliminate z, since on first glance equations #1 and #2 could be added to eliminate it.

[2] Add equations #1 and #2 to obtain an equation that only has x and y:

$$\begin{aligned} x + 2y - z &= 0 \quad ① \\ \underline{(+)\ 2x - y + z} &\underline{= 5} \quad ② \\ 3x + y &= 5 \quad ④ \end{aligned}$$

[3] Multiply equation #1 by 5 and add it to #3 to get a second new equation that also eliminates z.

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$$
\begin{array}{rcll}
5x + 10y - 5z & = & 0 & \textcircled{1} \times 5 \\
\underline{(+)\ 4x + 2y + 5z} & \underline{=} & \underline{6} & \textcircled{3} \\
9x + 12y & = & 6 &
\end{array}
$$

Divide through by 3 to make calculation easier.

$$
3x + 4y = 2 \quad \textcircled{5}
$$

[4] Solve the system made from equations #4 and #5:

$$
\begin{array}{rcll}
3x + y & = & 5 & \textcircled{4} \\
\underline{(+)\ -3x - 4y} & \underline{=} & \underline{-2} & \textcircled{5} \times -1 \\
-3y & = & 3 & \\
\mathbf{y} & \mathbf{=} & \mathbf{-1} &
\end{array}
$$

$$
\begin{array}{rcl}
3x + [-1] & = & 5 \\
3x & = & 6 \\
\mathbf{x} & \mathbf{=} & \mathbf{2}
\end{array}
$$

[5] Use one of the original equations (say, equation #1) to find z:

$$
\begin{array}{rcl}
x + 2y - z & = & 0 \\
[2] + 2[-1] - z & = & 0 \\
2 - 2 - z & = & 0 \\
\mathbf{z} & \mathbf{=} & \mathbf{0}
\end{array}
$$

[6] Check (x, y, z) = (2, −1, 0) in all three original equations.

$$
\begin{array}{rcl}
\textcircled{1}\ \text{LHS} = x + 2y - z & = & [2] + 2[-1] - [0] \\
& = & 2 - 2 - 0 \\
& = & 0 = \text{RHS} \checkmark
\end{array}
$$

$$
\begin{array}{rcl}
\textcircled{2}\ \text{LHS} = 2x - y + z & = & 2[2] - [-1] + [0] \\
& = & 4 + 1 + 0 \\
& = & 5 = \text{RHS} \checkmark
\end{array}
$$

$$
\begin{array}{rcl}
\textcircled{3}\ \text{LHS} = 4x + 2y + 5z & = & 4[2] + 2[-1] + 5[0] \\
& = & 8 - 2 + 0 \\
& = & 6 = \text{RHS} \checkmark
\end{array}
$$

NOTES

- The substitution method works similarly: Get an expression for one variable from one equation, and substitute it into both of the other two equations, and solve from there.
- You can use this strategy to solve systems of any size: eliminate variables and equations one by one until you have just one of each left. Then go backwards and evaluate for the variables that were removed in each step.

EXERCISES

A. Solve these systems:

1) $\begin{aligned} 2x + y + 2z &= 3 \\ -3x + 2y - z &= -3 \\ -4x + y - z &= -6 \end{aligned}$

2) $\begin{aligned} x - y + 2z &= 1 \\ 2x + y - 4z &= 4 \\ 3x + 2y - 2z &= 9 \end{aligned}$

3) $\begin{aligned} x + y + z &= 100 \\ x - y &= -10 \\ x - z &= -30 \end{aligned}$

4) $\begin{aligned} x + y &= 7 \\ 3y + 2z &= 9 \\ 4x &= 5 - z \end{aligned}$

5) $\begin{aligned} 3x + 5y &= -11 + 6z \\ 2x - y &= 8 + 3z \\ 4x - 3y &= 4 - 2z \end{aligned}$

6) $\begin{aligned} 2x + 3y - 5z &= 1 \\ x - y + \tfrac{5}{3}z &= \tfrac{1}{2} \\ 4x - 9y + 5z &= 0 \end{aligned}$

SOLUTIONS

A. (1) $(2, 1, -1)$ (2) $(2, 2, \tfrac{1}{2})$ (3) $(20, 30, 50)$ (4) $(2, 5, -3)$ (5) $(-1, -4, -2)$ (6) $(\tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{5})$