Recursive Sequences

Recall that a **sequence** is nothing more than a list of numbers. (If you are asked to add the numbers, that's when you have a series.) Any list of numbers is a sequence, but we've been looking at sequences generated by performing a calculation on the position number (usually denoted by n) to get the number for the list (denoted by a_n).

Let's look at how such a sequence is generated. For the sequence $a_n = 5n + 3$, the first element of the sequence, a_1, is $5(1) + 3 = 8$, the second one, a_2, is $5(2) + 3 = 13$, and so on. In this system we can jump ahead — we can find the fiftieth element, a_{50}, without needing to calculate the 49 elements in the sequence that come before it.

Another common way of defining a sequence is to use previous *terms* in the sequence to generate the following ones. Such a sequence might be defined like this: Start with 3. For each subsequent term, square the previous number and then subtract the previous number. Then the second term is $3^2 - 3 = 6$, the third term is $6^2 - 6 = 30$, the fourth is $30^2 - 30 = 870$, and so on.

A sequence like this is called a **recursive sequence**, with *recursive* meaning "referring back to itself". Because the sequence refers to its earlier terms, instead of being defined as a function in terms of n, it's defined in terms of previous values of a_n. The formula for the recursive sequence described above is:

$$a_1 = 3;\ a_n = (a_{n-1})^2 - a_{n-1}$$

The key to understanding formulas for recursive sequences is realizing that there are two variables used in defining it: a, which represents all the terms in the sequence and those terms are distinguished with subscripts, and n, which we use to count terms.

Let's say that we want to figure out the terms of a related sequence, identical to the last one, but with a different "seed number":

$$a_1 = 5;\ a_n = (a_{n-1})^2 - a_{n-1}$$

We know a_1 because it's given to us. (All recursive sequences have to start with one or more given numbers.) We want a_2, but to calculate it, we'll need to use the second part of the definition, after the semicolon. In order to make that equation read as "$a_2 = \cdots$" we need to set n to the value of 2:

$$a_2 = (a_{2-1})^2 - a_{2-1}$$

Now we have little arithmetic expressions in the subscripts, and we can simplify those:

$$a_2 = (a_1)^2 - a_1$$

The equation refers back to the value we have for $a_1 = 5$. Plug this in, and we're done:

$$a_2 = 5^2 - 5 = 20 \qquad \boxed{a_2 = 20}$$

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We can go on like this, calculating more terms:

$a_n = (a_{n-1})^2 - a_{n-1}$ We want a_3. Set $n = 3$.

$a_3 = (a_{3-1})^2 - a_{3-1}$

$= (a_2)^2 - a_2$

$= 20^2 - 20 = 400 - 20$

$a_3 = 380$

$a_n = (a_{n-1})^2 - a_{n-1}$ We want a_4. Set $n = 4$.

$a_4 = (a_{4-1})^2 - a_{4-1}$

$= (a_3)^2 - a_3$

$= 380^2 - 380 = 144{,}400 - 380$

$a_4 = 144{,}020$

…and so on. The formula is usually simple enough that we can make a "working rule" that lets us continue these calculations without needing to fiddle with the subscripts so much. It should be obvious that the next calculation would be $144{,}020^2 - 144{,}020$ without needing to resolve a_5's equation for its subscripts.

If we tried to write an equation for a_n in terms of n for these examples, it would be extremely complicated. The downside is that we lose the ability to calculate terms deep in the sequence directly — if we wanted to know a_{50} in this sequence, we *would* have to calculate all the terms in order up to a_{49} first.

THE FIBONACCI SEQUENCE

There is a famous recursive sequence that refers back to the previous two terms of the sequence:

$F_1 = 1,\ F_2 = 1;\ F_n = F_{n-1} + F_{n-2}$

You've seen this before, certainly, because every teacher covers it whenever recursive sequences come up in class, but let's go through the process for a couple of terms:

$F_1 = 1$

$F_2 = 1$

$F_n = F_{n-1} + F_{n-2}$ We want F_3. Set $n = 3$.

$F_3 = F_{3-1} + F_{3-2}$

$= F_2 + F_1$

$= 1 + 1$

$F_3 = 2$

$F_n = F_{n-1} + F_{n-2}$ We want F_4. Set $n = 4$.

$F_4 = F_{4-1} + F_{4-2}$

$= F_3 + F_2$

$= 2 + 1$

$F_4 = 3$

…and so on.

EXERCISES

A. Write the terms a_2 through a_5 for the following sequences.

1) $a_1 = 3; a_n = a_{n-1} + 5$
2) $a_1 = 6; a_n = a_{n-1} - 2$
3) $a_1 = 5; a_n = 2a_{n-1}$
4) $a_1 = 729; a_n = a_{n-1} \div 3$

B. Write non-recursive definitions for the sequences in section A. [*Hint:* you should recognize the structures of the sequences, now that you've written them down.]

C. Write the terms a_2 through a_5 for the following sequences.

1) $a_1 = 10; a_n = (a_{n-1})/2 + 3$
2) $a_1 = 2; a_n = 4a_{n-1} - 2$
3) $a_1 = 6; a_n = (a_{n-1})^2 \div 12$
4) $a_1 = 8; a_n = a_{n-1}(a_{n-1} - 5)$

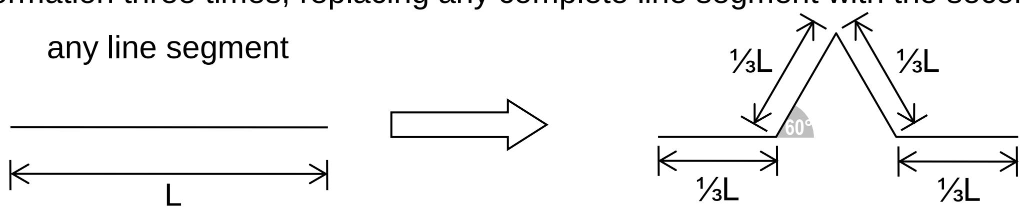
D. Write a recursive definition for each of these sequences. (There may be more than one correct way to do this.)

1) 10, 29, 86, 257, 770, 2309, 6926, …
2) 1000, 504, 256, 132, 70, 39, 23.5, …
3) 1, 11, 111, 1111, 11 111, 111 111, …
4) 3, 4, 9, 64, 3969, 15 745 024, … [*Hint:* Square numbers are involved]

E. Extend each of these recursive sequences to a_8.

1) $a_1 = 1, a_2 = 3; a_n = a_{n-1} + a_{n-2}$
2) $a_1 = 1, a_2 = 1, a_3 = 1; a_n = 4a_{n-1} - 2$
 $a_n = a_{n-1} + a_{n-2} + a_{n-3}$
3) $a_1 = 2, a_2 = 3; a_n = 5a_{n-1} - 2a_{n-2}$
4) $a_1 = 4, a_2 = 1, a_3 = 2, a_4 = 0;$
 $a_n = a_{n-1} - a_{n-3} + a_{n-4}$

F. Start with a horizontal line segment that is 3 cm long. Apply the following recursive transformation three times, replacing any complete line segment with the second shape:



SOLUTIONS

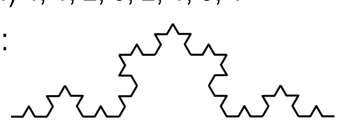
A: (1) 8, 13, 18, 23 (2) 4, 2, 0, −2 (3) 10, 20, 40, 80 (4) 243, 81, 27, 9

B: A1 and A2 are arithmetic sequences; A3 and A4 are geometric sequences.
(1) $a_n = 3 + 5(n - 1) = 5n - 2$ (2) $a_n = 6 - 2(n - 1) = -2n + 8$ (3) $a_n = 5 \cdot 2^{n-1}$
(4) $a_n = 729 \cdot (\frac{1}{3})^{n-1} = 2187 \cdot (\frac{1}{3})^n$

C: (1) 8, 7, 6.5, 6.25 (2) 6, 22, 86, 342 (3) 3, $\frac{3}{4}$, $\frac{3}{64}$, $\frac{3}{4096}$ (4) 24, 456, 205 656, 42 293 362 056

D: (1) $a_1 = 10; a_n = 3a_{n-1} - 1$ (2) $a_1 = 1000; a_n = (a_{n-1})/2 + 4$ (3) $a_1 = 1; a_n = 10a_{n-1} + 1$
(4) $a_1 = 3; a_n = (a_{n-1} - 1)^2$

E: (1) 1, 3, 4, 7, 11, 18, 29, 47 (2) 1, 1, 1, 3, 5, 9, 17, 31 (3) 2, 3, 4, 7, 6, 23, −16, 147
(4) 4, 1, 2, 0, 2, 1, 3, 1

F:  Look up "Koch snowflake" for more info.