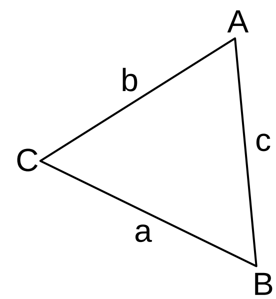
Law of Sines & Law of Cosines

With a right-angled triangle, we can use the trig ratios sine, cosine and tangent to figure out anything we might want to know about a triangle. We have two laws that tell us what to do in a triangle with no right angles. Which law we use depends on what we know about the triangle.

Recall that in labelling triangles, points (and therefore angles) get marked with capital letters, lengths of sides get marked with lower-case letters, and points and their opposite sides get marked with the same letter, as in the diagram here. Both laws assume that a triangle is labelled this way.

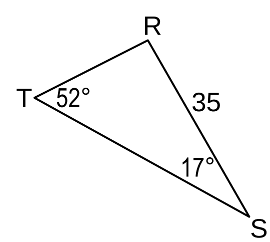


THE LAW OF SINES

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

You don't need all three fractions to use this law; pick whichever two work for you. On the other hand, you do need to know both numbers from one of the fractions. You need to know the measures of an angle-side opposite pair. If you don't, but you do know three numbers from the triangle, you have the information needed for the Law of Cosines. (This is how you can tell which Law to use.)

Example 1: Find the length of side s in the triangle at the right.



Solution: We know angle T is 52° and the side opposite it (the one side in the triangle which isn't used in drawing angle T) is 35 units, so we know T and t. We can use the Law of Sines to solve the problem.

$$\frac{\sin S}{s} = \frac{\sin T}{t}$$

$$t \sin S = s \sin T$$

$$s = \frac{t \sin S}{\sin T}$$

$$= \frac{35 \sin 17°}{\sin 52°} = 12.986\ldots \approx 13$$

The length of side s is 13 units. (This answer appears reasonable: smaller angles will be opposite smaller sides of the triangle. $\angle S < \angle T$ and therefore $s < t$.)

If we had been asked for side r, we could have first found angle R, since all three angles in any triangle add to 180°: R = 180 − 52 − 17 = 111°.

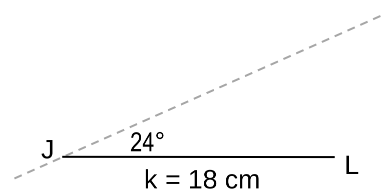
When a question asks you to solve for a side, there's no problem. When a question asks for an angle, under a particular set of circumstances, there might be.

Authored by Darren Rigby

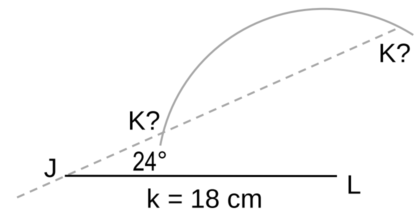
THE AMBIGUOUS CASE FOR THE LAW OF SINES

Example 2: In triangle JKL, ∠J = 24°, j = 11 cm and k = 18 cm. Find ∠K.

Solution: We don't have a diagram for this problem. Let's draw a perfectly accurate one. We can draw an exact 24° angle. The side k must be adjacent to ∠J and 18 cm long, but the problem doesn't give us any idea where to draw side j. Side j is the same thing as line segment KL, and all we know about K is that it's 11 cm away from L, somewhere on the dotted line:



We can use a compass to draw a circle centered on L with a radius of 11 cm, and see where it intersects the line. Our problem is that there are two such intersection points. The question does not give us enough information to choose between them.



What does the Law of Sines do for us here? We can solve the problem as before, and we get this:

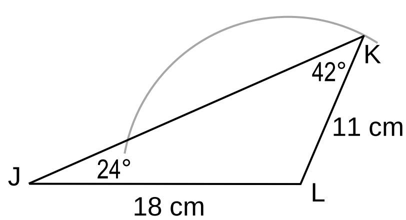
$$\frac{\sin J}{j} = \frac{\sin K}{k}$$

$$k \sin J = j \sin K$$

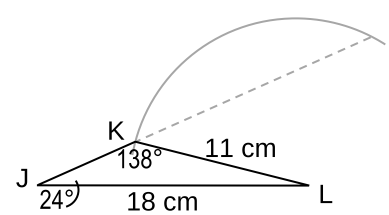
$$\sin K = \frac{k \sin J}{j}$$

$$= \frac{18 \sin 24°}{11} = 0.66557\ldots$$

$$K = \sin^{-1}(0.66557\ldots) = 41.726\ldots \approx 42°$$

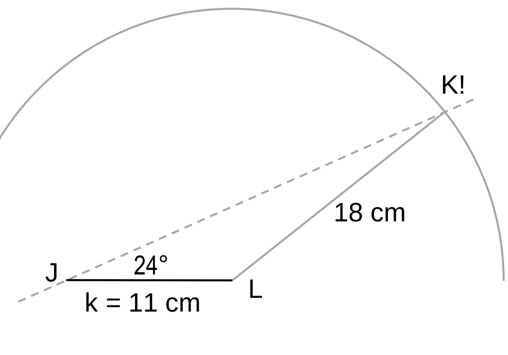


This answer describes the triangle on the left. It's a valid answer, but it's incomplete. The reason why is because we used the [$\sin^{-1}$] function on the calculator to get the angle. We would like $\sin^{-1}(x)$ to be a function, but it isn't one naturally. The problem is the same as the difference between the following two equations: $x^2 = 9$ and $x = \sqrt{9}$. They're clearly related, but the first equation has two solutions, and the second equation has only one. The difference is that we've decided to restrict the symbol $\sqrt{\ }$ to give only positive answers. Similarly, we've restricted $\sin^{-1}(x)$ to give only acute angles when it gets a positive value of x, and your calculator knows this, but the sine of an obtuse angle is also positive. In every ambiguous triangle, one solution is θ, where 0 < θ < 90°, and the other solution is 180° − θ. In our problem, then, the angle 180 − 41.726… = 138.274…° is also possible. You can check in your calculator that sin 138.274° = 0.66557… as well.



How can we tell that we have the ambiguous case? The given information must be side-side-angle around the triangle like Example 2 was (we were told j-k-J: a side, then the next side around the triangle, then the next angle). Also, the side opposite the angle we were given must be the shorter of the two sides we know about. Remember the sanity check from Example 1: smaller angles will be opposite smaller sides of the triangle.

Since 18 cm > 11 cm, all we know is ∠K > 24°, and both 42° and 138° work. If the sides were reversed, as in the diagram to the right, we don't have the ambiguous case. The unknown angle K would have to be smaller than 24°, so the obtuse angle doesn't work, and the diagram only creates one intersection point, and therefore only one possible triangle.



THE LAW OF COSINES

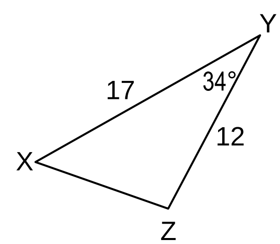
$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$b^2 = a^2 + c^2 - 2ac \cos B$$

$$c^2 = a^2 + b^2 - 2ab \cos C$$

We typically see three forms of the Law of Cosines, but they're all the same formula; the a, b, and c just rotate into new positions. The important part is that if "p^2" is isolated at the beginning, then the formula must end with "… cos P". No matter whether you're solving for the side length p or the angle ∠P, there is never an ambiguous case.

Example 3: Solve the triangle XYZ:



Solution: We haven't been given both an angle and the side opposite it, so we must use the Law of Cosines. The law involves all three sides, so we can use it to solve for the remaining side, y.

$$\begin{aligned} y^2 &= x^2 + z^2 - 2xz \cos Y \\ &= 12^2 + 17^2 - 2(12)(17) \cos 34° \\ &= 94.753\ldots \\ y &= \sqrt{94.753} = 9.7341\ldots \approx 10 \end{aligned}$$

With all three sides, we can use the Law of Cosines to get the other angles, but the Law of Sines is easier to use. We should also use the unrounded answer for y; otherwise the rounding errors will start to compound and propagate through the rest of the solution.

$$\frac{\sin Y}{y} = \frac{\sin X}{x} \Rightarrow x \sin Y = y \sin X \Rightarrow \sin X = \frac{x \sin Y}{y} = \frac{12 \sin 34°}{9.7341} = 0.68936\ldots$$

$$X = \sin^{-1}(0.68936) = 43.579\ldots \approx 44°$$

We know that this case isn't ambiguous because we used the medium side of the triangle for this calculation. Since z is the longest side, ∠Z is the largest angle, and it might be obtuse. ∠X is smaller than ∠Z, and if they were both obtuse, they'd both be more than 90°, and the sum of the angles in the triangle would be more than 180°, which is impossible. That means X really is about 44°. (This is a strategy. When solving for the second angle, target a smaller angle to avoid ambiguity.)

To get the last side, we don't need trigonometry at all. We can use the fact that all the angles in the triangle add up to 180°: 180° − 34° − 43.579° = 102.42…° ≈ 102°.

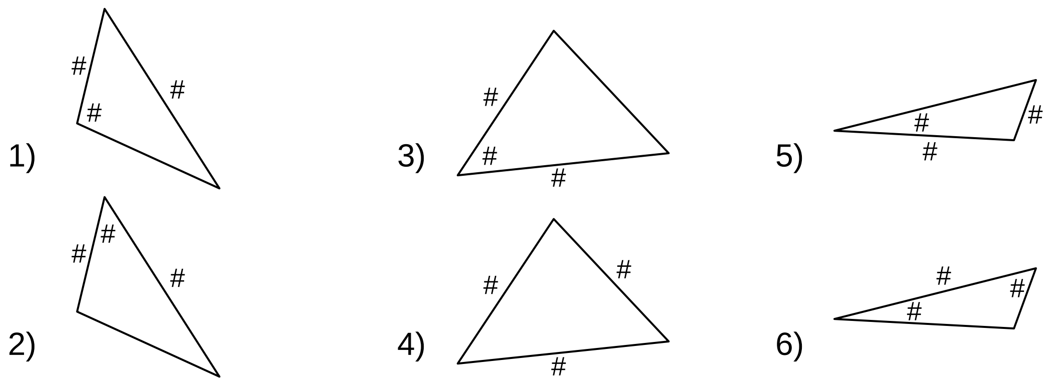
Bonus fact! If you were to use the Law of Cosines on a right triangle to solve for the hypotenuse, since cos 90° = 0, you get $c^2 = a^2 + b^2$. The Pythagorean Theorem is a special case of the Law of Cosines.

EXERCISES

A. Fill in the blanks: A question that yields two possible triangles when you use the Law of Sines will have its given information in the form _____-_____-_____ and the
side/angle side/angle side/angle

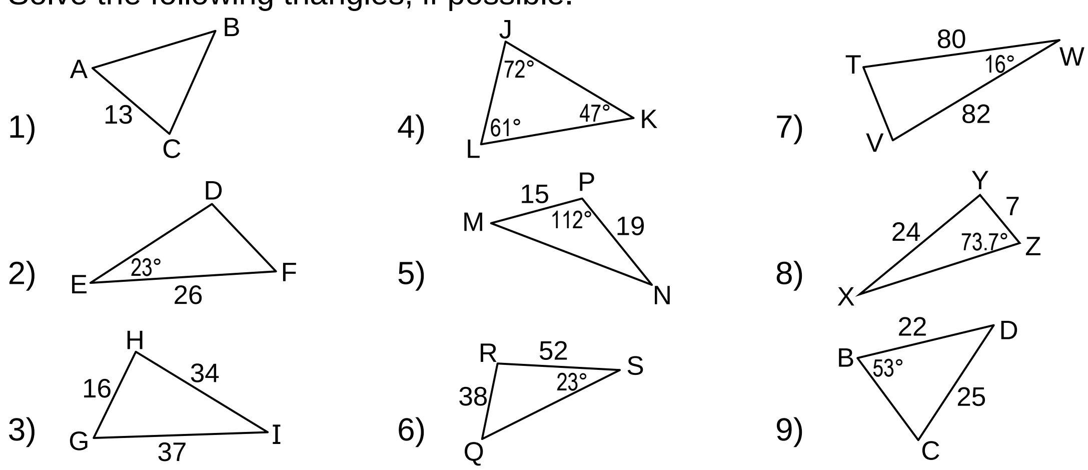
_________ side that we know will be opposite the angle that we know.
longer/shorter

B. In each of these triangles you're told three measurements (lengths of sides, measures of angles or both). These appear as "#". Which law can you use in each case?



C. Which one of the six problems in Section B would result in an ambiguous triangle?

D. Solve the following triangles, if possible.



SOLUTIONS

A: side-side-angle; shorter

B: The Law of… (1) Sines (2) Cosines (3) Cosines (4) Cosines (5) Sines
(6) Sines! You don't have enough information to use the Law of Cosines, but you can solve for the remaining angle since the three angles must add up to 180°.

C: B5 is ambiguous

D: (1) C = 67°, a = 14.43, c = 15.19 (2) D = 109°, e = 10.74, f = 20.44
(3) G = 66.7°, H = 87.7°, I = 25.6° (4) Not solvable—could be any size
(5) p = 28.28, M = 38.5°, N = 29.5° (6) Not solvable—ambiguous
(7) w = 22.63, T = 87.0°, V = 77.0° (8) X = 16.3°, Y = 90.0°, y = 25.01
(9) C = 44.7°, D = 82.3°, d = 31.02