Conic Sections:

The Ellipse

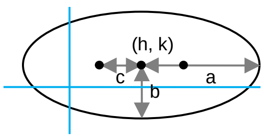
The equation for an ellipse has both an x^2 and y^2 term being added like a circle (rather than subtracted), but the coefficients in front of those terms do not match. The sizes of the coefficients determine whether the ellipse is "horizontal" or "vertical".

FORMULA FOR ELLIPSES

Once the formula for the ellipse is in standard form (described below), *a* and *b* are assigned by the convention that a > b, always.

Horizontal Major Axis:

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$$

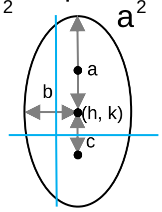


centre: (h, k)

length of major axis: 2a; length of minor axis: 2b
vertices: (h + a, k), (h − a, k)
foci: (h + c, k), (h − c, k), where $c^2 = a^2 - b^2$

Vertical Major Axis:

$$\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1$$



centre: (h, k)

length of major axis: 2a; length of minor axis: 2b
vertices: (h, k + a), (h, k − a)
foci: (h, k + c), (h, k − c), where $c^2 = a^2 - b^2$

Note: If, in general form, an equation has integer coefficients but is already equal to 1, then a and b are fractions. So $25x^2 + 49y^2 = 1$ can be rewritten as $\frac{x^2}{1/25}+\frac{y^2}{1/49} = 1$.

Example 1: Find the centre, vertices and foci of the ellipse $4x^2 + 8x + y^2 + 4y + 4 = 0$.

Solution: First we need to get the equation into the standard form. We start by completing the squares for x and for y.

$$\begin{aligned}
(4x^2 + 8x) + (y^2 + 4y) &= -4 \\
4(x^2 + 2x) + (y^2 + 4y) &= -4 \\
4(x^2 + 2x + 1 - 1) + (y^2 + 4y + 4 - 4) &= -4 \\
4(x^2 + 2x + 1) - 4 + (y^2 + 4y + 4) - 4 &= -4 \\
4(x + 1)^2 + (y + 2)^2 &= -4 + 4 + 4 \\
4(x + 1)^2 + (y + 2)^2 &= 4 \\
\frac{4(x+1)^2}{4}+\frac{(y+2)^2}{4} &= 1 \\
\frac{(x+1)^2}{1^2}+\frac{(y+2)^2}{2^2} &= 1
\end{aligned}$$

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Now we can see h, k, a and b: h = −1, k = −2, a = 2 and b = 1. (Remember, a must be greater than b.) Because a is in the y term, this is a vertical ellipse. The centre is at (h, k), or (−1, −2). The vertices are at (h, k ± a), or (−1, 0) and (−1, −4). We can calculate c for this ellipse:

$$\begin{aligned} c^2 &= a^2 - b^2 \\ &= 2^2 - 1^2 \\ &= 4 - 1 = 3 \\ \therefore c &= \sqrt{3} \end{aligned}$$

The foci, then, are at (h, k ± c), or (−1, −2 ± $\sqrt{3}$).

Example 2: Find the equation of the ellipse with vertices at (8, −2) and (−2, −2) and foci at (7, −2) and (−1, −2).

Solution: First, we must determine whether this is a horizontal or a vertical ellipse. The two vertices have the same y value, as do the foci, so this is a horizontal ellipse. (Vertical ellipses have the same x value for all four numbers.) The distance between the two vertices is equal to the major axis, which is 2a:

$$\begin{aligned} 2a &= \sqrt{[8-(-2)]^2 + [(-2)-(-2)]^2} \\ &= \sqrt{10^2 + 0^2} = 10 \\ \therefore a &= 5 \end{aligned}$$

The coordinate (8, −2) is the one that's farther to the right, so it must be (h + a, k). This means k = −2, and h + a = 8. Since a = 5, h must be 3. You can check by making sure the other coordinate is (h − a, k).

Finally we can get b by calculating the distance between the centre and either focus, which is c, and then using the relationship between a, b and c to get b:

$$\begin{aligned} c &= \sqrt{[3-(-1)]^2 + [(-2)-(-2)]^2} \\ &= \sqrt{4^2 + 0^2} = 4 \end{aligned}$$

$$\begin{aligned} c^2 &= a^2 - b^2 \\ b^2 &= a^2 - c^2 \\ &= [5]^2 - [4]^2 \\ &= 25 - 16 = 9 \\ \therefore b &= 3 \end{aligned}$$

We have h, k, a and b, so we can get the standard form of the equation of the ellipse:

$$\begin{aligned} \frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} &= 1 \\ \frac{(x-5)^2}{5^2} + \frac{(y+1)^2}{3^2} &= 1 \\ \frac{(x-5)^2}{25} + \frac{(y+1)^2}{9} &= 1 \end{aligned}$$

EXERCISES

A. Find the centre, vertices and foci for each ellipse:

1) $\frac{x^2}{4} + \frac{y^2}{9} = 1$

2) $\frac{(x-1)^2}{9} + \frac{(y-3)^2}{4} = 1$

3) $25x^2 + 4y^2 = 100$

4) $4x^2 + 36y^2 = 1$

5) $4x^2 - 8x + 16y^2 + 64y + 4 = 0$

6) $x^2 - 6x + 9y^2 - 108y + 315 = 0$

B. Find the equation of an ellipse with the following features:

1) vertices: (3, 0), (−3, 0); endpoints of minor axis: (0, 2), (0, −2)

2) vertices: (1, 5), (1, 1); foci: (1, 4), (1, 2)

3) foci: (9, 4) , (9, 28); endpoints of minor axis: (4, 16), (14, 16)

C. Graph the ellipse from Exercise A5.

SOLUTIONS

A:

(1) centre: (0, 0); vertices: (0, 3), (0, −3); foci: $(0, \sqrt{5})$, $(0, -\sqrt{5})$

(2) centre: (1, 3); vertices: (4, 3), (−2, 3); foci: $(1 + \sqrt{5}, 3)$, $(1 - \sqrt{5}, 3)$

(3) centre: (0, 0); vertices: (0, 5), (0, −5); foci: $(0, \sqrt{21})$, $(0, -\sqrt{21})$

(4) centre: (0, 0); vertices: (½, 0), (−½, 0); foci: $(\frac{\sqrt{2}}{3}, 0)$, $(-\frac{\sqrt{2}}{3}, 0)$

(5) centre: (1, −2); vertices (5, −2), (−3, −2), foci: $(1 + 2\sqrt{3}, -2)$, $(1 - 2\sqrt{3}, -2)$

(6) centre: (3, 6); vertices $(3 + 3\sqrt{2}, 6)$, $(3 - 3\sqrt{2}, 6)$, foci: (7, 6), (−1, 6)

B: (1) $\frac{x^2}{9} + \frac{y^2}{4} = 1$ (2) $\frac{(x-1)^2}{3} + \frac{(y-3)^2}{4} = 1$ (3) $\frac{(x-9)^2}{25} + \frac{(y-16)^2}{169} = 1$

C: