Probability 2:
Compound Events

Events that are **independent** of each other are events whose outcomes cannot affect each other. This isn't a precise definition — how can you tell whether events are independent if they involve the same people, for example? — but it will do for now. You'll get a better definition and a test for independence later in your course.

The Probability 1 worksheet discussed how to determine the probability of one random event. If you have a series of independent events, you can calculate the probability of the series from those individual events.

Example 1: Three coins are flipped: a penny, a nickel and a dime. What is the probability of getting heads on (a) the penny? (b) the nickel? (c) the dime? (d) all three?

Solution: There are two outcomes for each flip, and those outcomes are equally likely, so we can say that the answer to (a), (b) and (c) is ½. (d) is a harder question. We go back to our definition of how to calculate the probability of a random event: For outcomes that are equally likely,

$$P(\text{event}) = \frac{\text{the total number of successes}}{\text{the total number of outcomes}}$$

We can list all the outcomes of the coin flips. There are eight of them, and only one of them is a success as the question defines it, so the answer is ⅛.

p	n	d
H	H	H
H	H	T
H	T	H
H	T	T
T	H	H
T	H	T
T	T	H
T	T	T

This answer is also equal to ½ × ½ × ½, the product of the answers to (a), (b) and (c). This suggests the rule:

> *The probability of a series of independent events is equal to the product of the probabilities of the individual events.*

or: P(A and B) = P(A) · P(B), where A and B are independent.

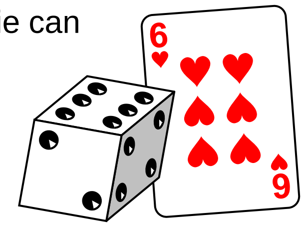
This is a different rule from the addition rule in the Probability 1 worksheet! The addition rule deals with <u>one event</u>, and multiple outcomes of that event. The multiplication rule here refers to the outcomes of <u>a series of events</u>.

We use "and" and "or" to describe combinations of events, but they're common words used in many contexts. It may help you to think in terms of "*and then*" versus "*or else*".

If the parts of the problem can be connected with "and then" — I flip the penny *and then* I flip the nickel — then every piece of the problem is another hoop to jump through, and it only gets harder to succeed. We multiply these probabilities (since multiplying fractions makes them smaller), and we check if the events are *independent*.

If the parts of the problem can be connected with "or else" — I flip a head on the penny *or else* I roll an even number on a die — then every piece of the problem is a way to succeed, and it gets easier. We add these probabilities (since adding fractions makes them bigger), and we check if the events are *disjoint*. (See Probability 1 for a review.)

Example 2: What is the probability of rolling a 6 on a standard die, and then drawing a 6 from a well-shuffled deck of cards?



Solution: These events are independent — nothing we do with the die can affect the cards — so we can use the rule about a series of events.

P(⚅) = $\frac{1}{6}$

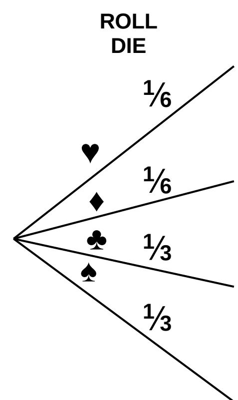
P([6?]) = $\frac{4}{52} = \frac{1}{13}$

P(⚅ and [6?]) = $\frac{1}{6} \times \frac{1}{13} = \frac{1}{78}$

Example 3: A deck of cards has had some cards removed. It now contains 12 hearts (♥), 9 diamonds (♦), 6 clubs (♣) and 3 spades (♠), A special six-sided die has 1 heart, 1 diamond, 2 clubs and 2 spades as its faces. What is the probability of (a) rolling a heart on the die and drawing a heart from the deck? (b) rolling a suit on the die and drawing the same suit from the deck?

Solution: (a) On the die, there is 1 heart out of six sides, so P(roll [♥]) is $\frac{1}{6}$. In the deck there are 12 hearts out of 12 + 9 + 6 + 3 = 30 cards, so P(draw [?♥] = $\frac{12}{30} = \frac{2}{5}$.
∴ P(roll [♥] and draw [?♥]) = $\frac{1}{6} \times \frac{2}{5} = \frac{2}{30} = \frac{1}{15}$.

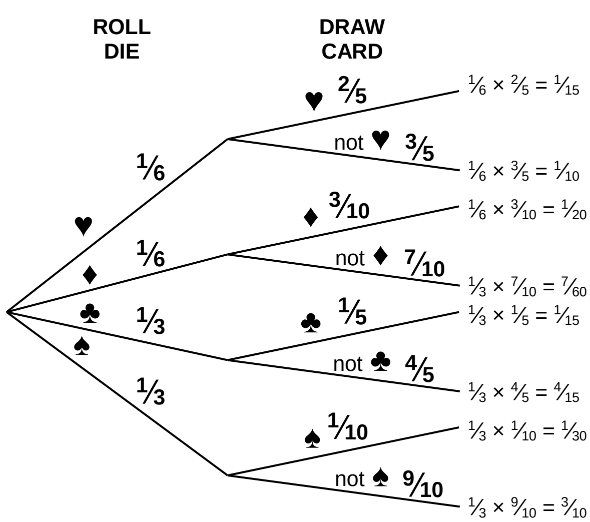
(b) This is a more complex question. Every outcome on the die might result in a success in the sequence of events. It will be easier to keep track of our information with a **tree diagram**. For each event, we add a level of branches to the tree and each outcome gets its own branch. Each branch also gets labelled with the probability of that outcome. We can calculate the total probability for the question by adding the probability of each branch that results in a success.



We'll start with the roll of the die. There are four outcomes for our purposes, so we draw four branches in the first level.

P(roll [♥]) = P(roll [♦]) = $\frac{1}{6}$ and P(roll [♣]) = P(roll [♠]) = $\frac{1}{3}$

Each of these outcomes will have two further outcomes when we draw a card: the suits match or they don't. The probabilities of matching each suit are different, so we must calculate each one separately. We'll also be using the rule that says P(not A) = 1 − P(A):



The tree has eight "leaves": eight endings to the branches that list the possibilities. If you were to add up the probabilities for all eight leaves, the sum would be 1.

Four of the leaves represent successes in this experiment: the first, third, fifth and seventh. If we add up those four fractions, we get the probability of rolling a suit on the die, and then drawing the same suit from the deck. ($\frac{1}{15} + \frac{1}{20} + \frac{1}{15} + \frac{1}{30} = \frac{13}{60}$) (While we used "and then", we're not being very specific — rolling "a suit" isn't a single event here, and we cannot calculate the probability by multiplying alone. To describe this event fully, we'd have to say "rolling a heart and then drawing a heart, *or else* rolling a diamond and then drawing a diamond…". The full description is quite complex, which is why we use a tree diagram. The diagram makes it easier to keep track.)

INDEPENDENCE

For events like drawing a card and rolling a die, it's obvious whether or not the events are independent. They clearly have nothing to do with each other. Events in real life are usually not so clear. We test for statistical independence by seeing whether events follow the rule we stated earlier: P(A and B) = P(A) × P(B) when events are independent. If we know the probability of two events by themselves, and the probability that they happen at the same time, then we can use the rule backwards to see if the events are independent.

Example 4: Brenda the batter and Peter the pitcher have played many baseball games together. Out of 400 at-bats when Peter was pitching, Brenda got to first base (or farther) 244 times. In those same games, Peter used his patented fastball one quarter of the time. He's noticed that when he hasn't used his fastball, he has struck Brenda out or she didn't get to base 117 times. Is Peter's fastball effective when Brenda is at bat?

Solution: First we need to record our information. We use a **two-way table**, also known as a **contingency table**.

		BRENDA BATTING		
		got to base	struck out	
PETER PITCHING	other	183	117	300
	fastball	61	39	100
		244	156	400

The total number of trials (games) is 400. Brenda got to base 244 times, so that's the total under the "Got to base" column. Peter used his fastball one-quarter of the time, or 100 times, so that's the total on the "Fastball" row. Finally, when Peter didn't use his fastball, Brenda struck out 117 times. That number goes in the box where "Other" and "Struck out" intersect.

With this information, we can fill in the rest of the data, since the rows and columns have to add up. The finished table is at left.

If Peter's fastball is effective against Brenda, then she strikes out more often when he uses it. This would mean that the two events would be dependent: Peter using his fastball changes Brenda's probability of striking out. We need to test these variables for independence. We'll use S for Brenda striking out, and F for Peter using his fastball as the two events. According to the compound event rule:

$$P(S \text{ and } F) = P(S) \times P(F) \text{ if and only if S and F are independent}$$

$$P(S \text{ and } F) = \tfrac{39}{400} = 0.0975 \qquad P(S) \times P(F) = \tfrac{156}{400} \times \tfrac{1}{4} = \tfrac{156}{1600} = 0.0975$$

The answers are the same, so these events are independent. Peter's fastball isn't particularly effective against Brenda, but it's also no worse than anything else he uses, on average.

EXERCISES

A. For each compound event, should the probabilities of the parts in bold be added together ("or else") or multiplied together ("and then")?

1) Pick a random VCC student and get a **music student** or a **fashion student**.
2) On a die, roll a square number. The square numbers on a die are **1** and **4**.
3) Pull 2 socks, **one** after **the other**, from a drawer that are either brown or black.
4) Spin **more than $500 on a wheel** and pick a **letter that's in a word puzzle**.

B. Find the probabilities of the following compound events. Assume in all cases that the events are both independent and random. Express your answer as a fraction:

1) flipping a penny and getting tails and flipping a dime and getting heads
2) drawing a jack, queen or king from a deck of cards and rolling a 5 on a die
3) picking a date from the past that is a Tuesday in October (ignoring leap years)
4) rolling an odd number on a red die and rolling a multiple of 3 on a blue die
5) Choosing "B.C." from a list of Canadian provinces and "Nunavut" from a list of Canadian territories
6) choosing a white ball from box #3 below, rolling a 6 on a pair of dice, and choosing a red crayon from a box of 8 different colours (including red, of course)

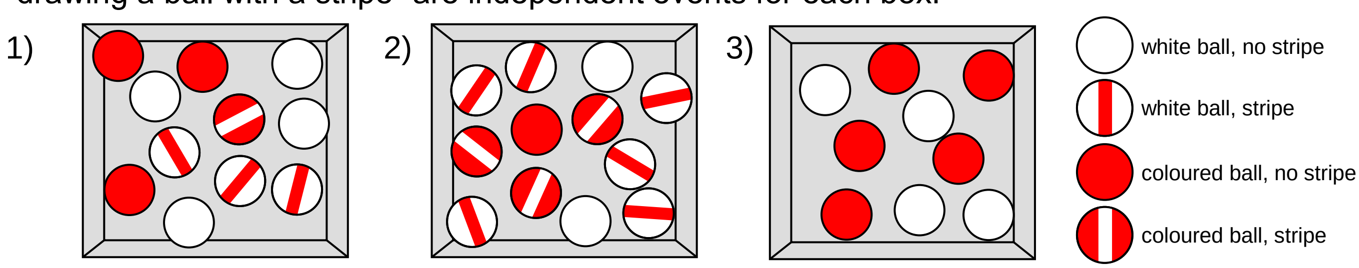
C. Consider three fair six-sided dice with the following numbers on their faces:

A: 2, 2, 2, 4, 9, 9 B: 1, 1, 6, 6, 8, 8 C: 3, 3, 3, 5, 5, 7

Use tree diagrams to find the probability that I roll the higher number if:

1) you choose die A and I choose die C, and we roll
2) you choose die B and I choose die A, and we roll
3) you choose die C and I choose die B, and we roll

D. In the boxes below, there are white balls and coloured balls, some with stripes and some without stripes. (A white ball with a stripe is still considered a white ball.) One ball from each box is drawn at random. Determine whether "drawing a white ball" and "drawing a ball with a stripe" are independent events for each box:



SOLUTIONS

A: (1) added (music or else fashion) (2) added (1 or else 4) (3) multiplied (dark sock and then a dark sock) (4) multiplied (spin $500 or more and then pick a letter)

B: (1) $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$ (2) $\frac{12}{52} \times \frac{1}{6} = \frac{1}{26}$ (3) $\frac{1}{7} \times \frac{31}{365} = \frac{31}{2555}$ (4) $\frac{3}{6} \times \frac{2}{6} = \frac{1}{6}$
(5) $\frac{1}{10} \times \frac{1}{3} = \frac{1}{30}$ (6) $\frac{5}{9} \times \frac{5}{36} \times \frac{1}{8} = \frac{25}{2592}$

C: (1) $\frac{7}{12}$ (2) $\frac{5}{9}$ (3) $\frac{11}{18}$ No matter which die you pick, I win!

D: (1) P(white) × P(stripe) = $\frac{7}{11} \times \frac{4}{11} = \frac{28}{121}$; P(white with stripe) = $\frac{3}{11}$; not independent
(2) P(white) × P(stripe) = $\frac{8}{12} \times \frac{9}{12} = \frac{1}{2}$; P(white with stripe) = $\frac{1}{2}$; independent
(3) P(white) × P(stripe) = $\frac{4}{9} \times 0 = 0$; P(white with stripe) = 0; independent