Combinatorics

The field of **combinatorics** is the mathematics of counting. Combinatorics is used to count large quantities of things in a systematic way, which is not an easy or trivial task.

Before describing the sorts of problems we face in combinatorics, we should define a new mathematical symbol: n**!** We read this as "n **factorial**". The factorial symbol *only applies to whole numbers*, and n! indicates that we multiply together all the numbers from n down to 1. So 4! = 4 × 3 × 2 × 1 = 24. We also define **0! = 1** because combinatorics problems only work out correctly when we use this value.

An important skill in working with factorials is learning how to cancel them. Because factorials count down to 1, when we divide factorials many of the factors can be cancelled which simplifies the evaluation of the expression:

$$\frac{10!}{7!} = \frac{10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1} = 10 \times 9 \times 8$$

$$\frac{(n+2)!}{n!} = \frac{(n+2) \times (n+1) \times n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1}{n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1} = (n+2) \times (n+1)$$

> In general, for n > k: $\dfrac{n!}{k!}$ = n × (n − 1) × (n − 2) × … × (k + 1)

PERMUTATIONS

A **permutation** is an arrangement of items, without any item repeating, where the order of the items matters. Examples include the letters in a word, the digits in a number, or a committee where everyone has a title (a president is different from a secretary or a treasurer).

We can solve permutation problems using the "blanks" method. In this method we draw a blank for every position we want to fill, then write how many ways each position can be filled for every blank, and then multiply the resulting numbers together.

Example 1:   How many five-letter sequences can be made from the letters P, Q, R, S and T, without using any letter more than once?

Solution: This is a permutation, since the order of the letters matters. Since we are making a five-letter sequence, we draw five blanks. There are five letters available to fill the first blank, so we write "5" there. For the second blank, one letter is gone (though we don't know which one), and there are four left to pick from, so write "4" in the second blank. If we continue with this logic, we get:

$$\underline{\;5\;}\quad\underline{\;4\;}\quad\underline{\;3\;}\quad\underline{\;2\;}\quad\underline{\;1\;}$$

Now we multiply those numbers. 5 × 4 × 3 × 2 × 1 = 120. This is the answer.

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Notice, too, that this is equal to 5 factorial. We write this as $_5P_5 = 5!$, and in general **$_nP_n = n!$**.

Example 2:   How many *three*-letter sequences can be made from the letters P, Q, R, S and T, without using any letter more than once?

Solution: We can use blanks here as well. We need a three-letter sequence, so we write three blanks, and fill them with the number of letters that could go in each blank at each stage of the problem:

$$\underline{\;5\;}\quad\underline{\;4\;}\quad\underline{\;3\;}$$

We multiply the numbers: $5 \times 4 \times 3 = 60$. We write this as $_5P_3$ (out of 5 items, pick 3 of them). The general form of this problem is:

$$\boxed{_nP_k = \frac{n!}{(n-k)!}\text{, where } n \geq k, \text{ and } n, k \text{ are whole numbers.}}$$

We solve any permutation problem with identical items like this:

$$\boxed{\text{The number of ways of arranging n items including k identical items is } \frac{n!}{k!}.}$$

Example 3:   How many five-letter sequences can be made from the letters Q, Q, R, S and T, without using any letter more than once?

Solution: This problem looks similar to Example 1, but now we have 2 Q's. The Q's are identical, so the answer is $\frac{5!}{2!} = 120 \div 2 = 60$.

We can extend this idea to multiple sets of identical items. The number of five-digit sequences we can make out of the digits 1, 1, 2, 2, 2 is $\frac{5!}{2! \cdot 3!} = 120 \div (2 \times 6) = 10$. We divide by 2! and 3! because we have a set of two identical 1's and three identical 2's.

COMBINATIONS

A **combination** is a collection of items where the order of the items *does not* matter. Examples include cards in a poker or bridge hand (since the order the cards were dealt to you is irrelevant in most card games), or a committee where no one has a title (either a person is on the committee or not).

We write combinations in two ways. The number of ways of taking 3 items from a group of 10 without order is either $_{10}C_3$ or $\binom{10}{3}$, and both are read "10 choose 3". It is important to notice that $\binom{10}{3}$ is not a fraction! The fraction $^{10}\!/_3$ is not a whole number, but 10-choose-3 is the number of ways of doing something, and that must be a whole number.

We calculate combinations like this:

$$\boxed{_nC_k = \binom{n}{k} = \frac{n!}{k!(n-k)!}\text{, where } n \geq k, \text{ and } n, k \text{ are whole numbers.}}$$

Example 4: How many ways can an organization with 10 people form (a) a four-person committee? (b) a six-person committee?

Solution: This is a combination, since no one on the committee has a position.

(a) $_{10}C_4 = \dfrac{10!}{4!(10-4)!} = \dfrac{10!}{4! \cdot 6!} = \dfrac{10 \times 9 \times 8 \times 7 \times \cancel{6!}}{4! \cdot \cancel{6!}} = \dfrac{10 \times \cancel{9}^3 \times \cancel{8}^2 \times 7}{\cancel{4} \times \cancel{3} \times \cancel{2} \times 1} = 10 \times 3 \times 7 = 210$

(b) $_{10}C_6 = \dfrac{10!}{6!(10-6)!} = \dfrac{10!}{4! \cdot 6!} = 210$

Notice that the two answers are the same. This is because we're dividing the group of 10 into a group of 4 and a group of 6 either way.

Example 5: How many ways can 6 men and 4 women form a committee of 2 men and 2 women?

Solution: This sounds like a single combination problem, but it's actually two: how many ways can you choose 2 out of 6 men and then 2 out of 4 women? That's $_6C_2$ and $_4C_2$, respectively, and we can make them into one answer by multiplying them. This idea of putting pieces of combinatorics problems together with multiplication is called the **Fundamental Counting Principle**: *If there are n_1 ways of doing a first step in a process, and n_2 ways of doing the second step, and so on, then the number of ways of doing the whole process is $n_1 \times n_2 \times \ldots$*

So for our question, $_6C_2 \times {_4C_2} = \dfrac{6!}{2! \cdot 4!} \times \dfrac{4!}{2! \cdot 2!} = 15 \times 6 = 90$

OTHER PROBLEMS

There are combinatorics problems that are neither permutations nor combinations. For most of these, we can use the blanks method to get the answer.

Example 6: How many four-digit numbers can be made from the digits 0–9 if (a) the digits may not be repeated and the number cannot start with 0? (b) any digit can be repeated any number of times, and numbers can start with 0?

Solution: These may sound like permutation problems, but part (a) has a restriction and part (b) allows you to reuse numbers. Neither of these problems can be solved using the formulas presented in the section on permutations.

(a) We draw four blanks. There are ten digits, but the first blank can only have nine (the 0 is restricted). After that there are no restrictions, so the second blank has nine possibilities, and so on. The answer is $\underline{9} \times \underline{9} \times \underline{8} \times \underline{7} = 4536$. For this kind of problem, *deal with all restrictions first*. You don't have to fill in the blanks in order from left to right!

(b) We draw four blanks. For all four of them, all ten digits are available. The answer, then, is $\underline{10} \times \underline{10} \times \underline{10} \times \underline{10} = 10,000$. In this problem, each position in the number has ten "states". In general, whenever there are n things, each of which can be in k states with repetition, there are $\mathbf{k^n}$ possibilities.

EXERCISES

A. Simplify (but do not evaluate) these factorial expressions.

1) $\dfrac{7!}{3!}$
2) $\dfrac{n!}{(n-2)!}$
3) $\dfrac{(x+3)!}{(x+1)!}$
4) $\dfrac{(k+1)!}{(k-1)!}$

B. Are these situations permutations, combinations or neither?
1) A child makes a stack out of five blocks which are all different colours.
2) A poker player is dealt five cards from a well-shuffled deck.
3) A cat taps on the keys of a piano, playing six notes.
4) A safe manufacturer chooses three distinct numbers to form a combination.
5) A person flips four different coins and looks to see whether they're heads or tails.

C. Determine the number of possibilities in these permutation problems.
1) A valet parks 6 cars in the 6 spaces farthest from a restaurant.
2) A baseball coach decides in what order his 9 players will bat.
3) An official draws the names of a grand prize winner and a runner-up from a hat containing 15 names.
4) A valet parks 6 cars in the 10 spaces farthest from a restaurant. [*Hint: Empty parking spaces are indistinguishable.*]

D. Determine the number of possibilities in these combination problems.
1) A shopper takes 2 different flavours of chips from the 7 available in the store.
2) A captain chooses 4 teammates for dodgeball out of 8 players.
3) A customer orders a sundae by choosing 2 ice cream flavours out of 10 and 3 toppings out of 5.

E. Determine the number of possibilities in these problems.
1) The 10 students in a class each raise their hands or don't when the teacher asks a question.
2) A four-letter "word" is formed from the letters A, B, C, D and E, but the word cannot start or end with the letter C. Letters may be repeated.
3) A four-letter "word" is formed from the letters A, B, C, D and E, but the word cannot start or end with the letter C. Letters may <u>not</u> be repeated.
4) An three-digit area code is formed, where the first digit may not be 0 or 1, the second digit must be 0 or 1, and the last digit may be anything.

SOLUTIONS

A. (1) $7 \times 6 \times 5 \times 4$ (2) $n(n-1)$ (3) $(x+3)(x+2)$ (4) $(k+1)k$
B. (1) permutation (2) combination (3) neither (4) permutation (despite its name!)
   (5) neither
C. (1) $_6P_6 = 720$ (2) $_9P_9 = 362{,}880$ (3) $_{15}P_2 = 210$ (4) $10! / 4! = 151{,}200$
D. (1) $_7C_2 = 21$ (2) $_8C_4 = 70$ (3) $_{10}C_2 \times _5C_3 = 450$
E. (1) $2^{10} = 1024$ (2) $4 \times 5 \times 5 \times 4 = 400$ (3) $4 \times 3 \times 2 \times 3 = 72$ (4) $8 \times 2 \times 10 = 160$