# Transformations of Graphs

Graphs can come in families. All parabolas that point up or point down, for example, are in the same family, and all of their equations come from the equation $y = x^2$. This worksheet will show you how to alter one of the basic graphs, like $y = x^2$, so a new graph can be drawn without having to start over.

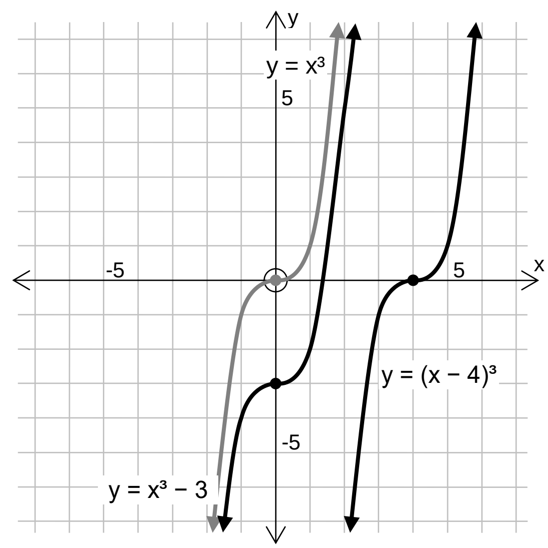
If a basic graph (such as $y = x^2$, or $y = \sin x$, or $y = 1/x$, …) is $y = f(x)$, then the simplest transformations come in the form:

$$y = af\left(\frac{x-h}{b}\right) + k$$

The explanations of the new variables are below.

**TRANSLATIONS (Shifts/Slides)**

h and k tell you how much to shift the basic graph without rotating it. h is the amount of horizontal shift and k is the amount of vertical shift.




The graph at the right starts with a basic graph, $y = x^3$, in grey. Below this graph, we see the graph of $y = x^3 - 3$. In this case k is $-3$. (It's not h since the 3 is not being cubed in the equation.) This means the graph does not change shape, but moves down 3 units. The marked point at origin in the grey graph has moved to $(0, -3)$.

To the right of the original graph is $y = (x - 4)^3$. Here $h = 4$, not $-4$. (If h were $-4$, the equation would be $(x - [-4])^3$, or $(x + 4)^3$.) The marked point at origin in the grey graph has moved to $(4, 0)$. Notice in the original transformation equation above, h is subtracted. It's a general rule that when a number affects x directly, inside the basic function, the transformation behaves the opposite of how it looks. The expression "$y = (x - 4)^3$" *looks* like it should move the graph 4 units to the negative side of the x-axis, but instead it does the opposite and goes to the positive side. On the other hand, anything that affects the function as a whole, such as adding or subtracting a number after we calculate $x^3$ in this example, affects the y-value directly and *does* behave as it looks like it should. ($y = x^3 - 3$ looks like it should move vertically 3 toward the negative side of the y-axis, and it does.)

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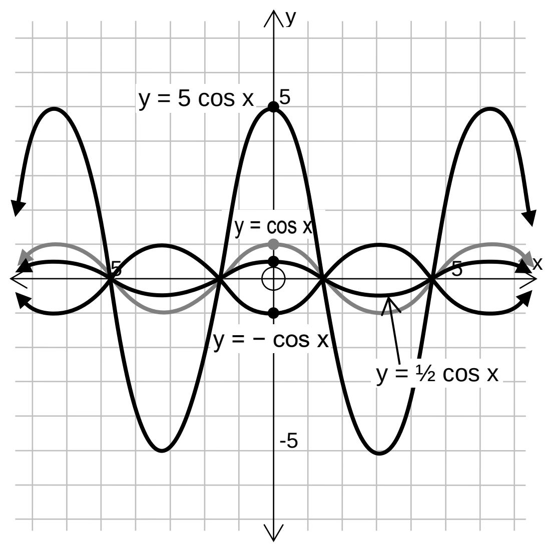
### VERTICAL STRETCHING AND SHRINKING

The "*a*" at the beginning of the transformation equation gives us the degree of "stretch" for the new graph. We multiply the y-coordinate of each point in the basic graph by *a*.

In this example the grey curve is the basic curve y = cos x. There are three other curves in the graph, all from the same family.




y = 5 cos x has been stretched to 5 times the size of the original curve, both top and bottom. When |a| > 1, the curve gets stretched.

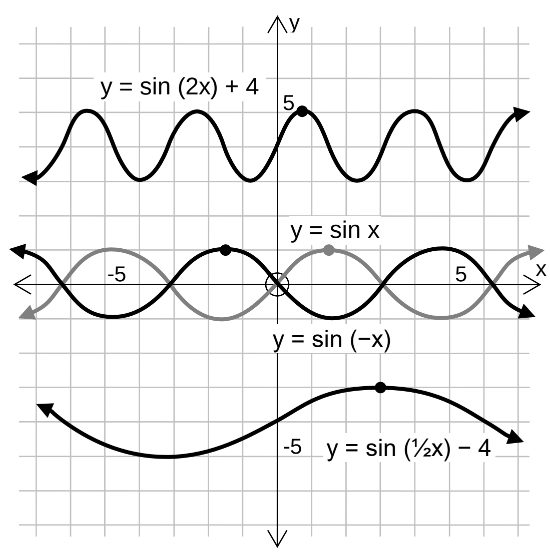
y = ½ cos x has been shrunk to ½ the height of the original curve. When 0 < |a| < 1, the curve shrinks.

y = − cos x has been reflected through the x-axis. Any negative value for the variable *a* reflects the graph through the x-axis in addition to possibly stretching or shrinking it.

Notice these transformations do what they look like they should do: y = 5 cos x takes y = cos x and multiplies it by 5 rather than by ⅕, and because they affect the cosine, and not x itself, these transformations affect y.

### HORIZONTAL STRETCHING AND SHRINKING

The b acting as the denominator inside the brackets in the transformation equation tells us what number to multiply the coordinates of each point of the function by when we plot the new graph. The basic curve here is y = sin x.




The bottom graph is y = sin (½x) − 4, or $y = \sin \frac{x}{2} - 4$. It has been shifted down 4 so it is clearer to see. The x within the sine function has been divided by 2, and according the rule about changes affecting x directly, the graph does the opposite of what is expected: it gets twice as wide.

Similarly, the graph of y = sin (2x) + 4 looks like it gets multiplied by two, but the waves are half as long instead. (Consider: $2x = \frac{x}{½}$, so the b in our transformation equation is ½.)

For the third equation, y = sin (−x), the curve has been reflected through the y-axis. Any negative value for b reflects the graph through the y-axis in addition to possibly stretching or shrinking it.

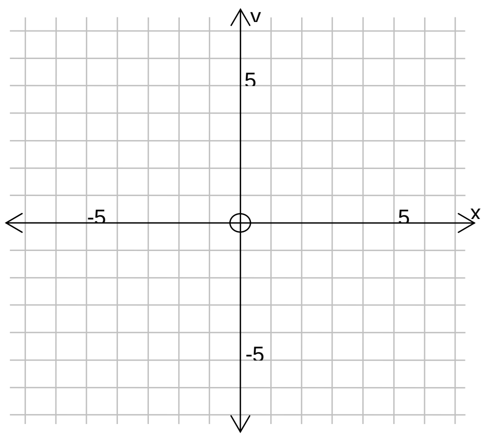
## FINAL NOTES

When working with parabolas, with a basic equation of $y = x^2$, we don't concern ourselves with horizontal stretching since the equation can be rewritten so any horizontal stretch is converted to vertical stretching. All stretching is done with respect to the x- and y-axes, so that the intercepts stay the same, and the rest of the graph spreads out away from them, or pulls in toward them.
When there are multiple transformations in a single equation, do any **horizontal /vertical stretching** first, then any **horizontal/vertical shifting**.

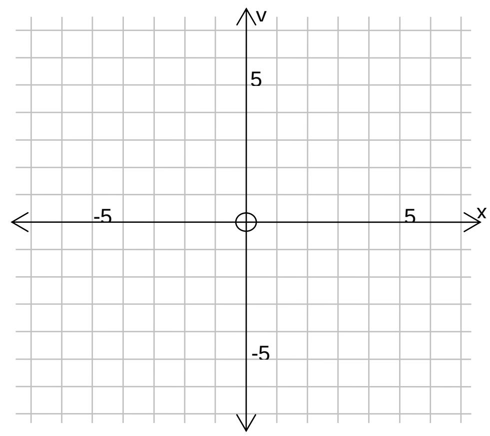
## EXERCISES

A. For each set of equations, determine the basic function for the set, and then sketch the graph of each equation in the set.

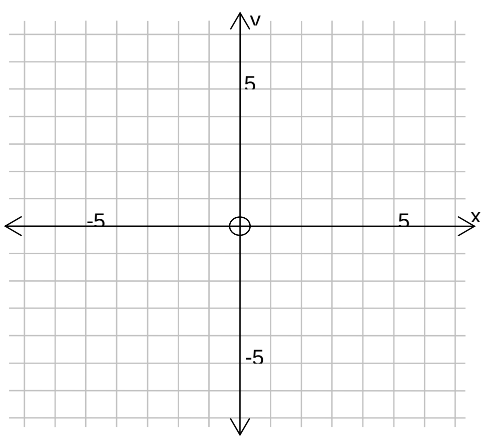
1) (a) $y = x^2 + 3$ (b) $y = (x + 3)^2$
   (c) $y = (3x)^2$ (d) $y = 9 \cdot x^2$



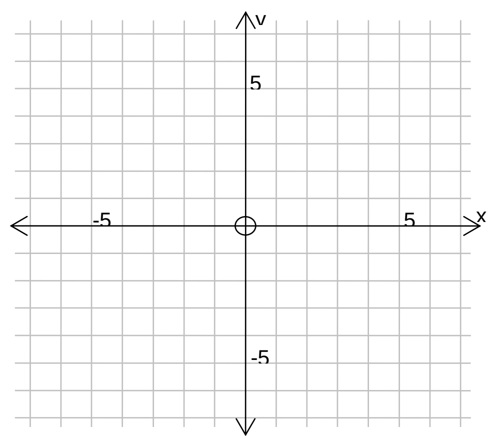

2) (a) $y = |x| + 4$ (b) $y = |3x|$
   (c) $y = -|\tfrac{1}{3}x - 2|$ (d) $y = -|\tfrac{1}{3}x| - 3$




3) (a) $y = \sin(\tfrac{1}{2}x)$ (b) $y = 2 \sin x$
   (c) $y = -\sin x + 6$ (d) $y = \sin(-x) + 6$

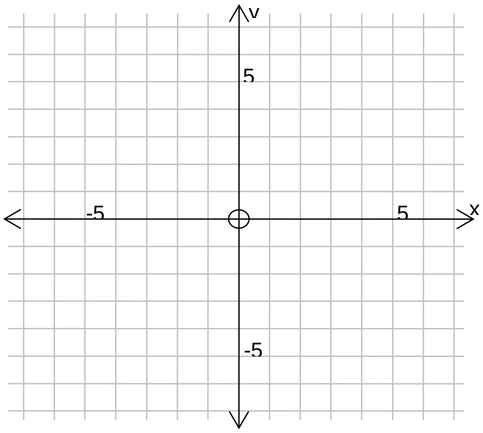



4) (a) $y = \sqrt{x} + 1$ (b) $y = -\sqrt{x} + 1$
   (c) $y = \sqrt{-x} + 1$ (d) $y = \sqrt{3x - 2} + 4$



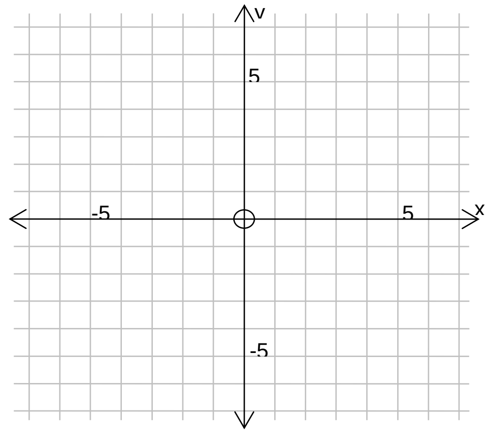
5) (a) $y = \frac{1}{3x+2}$ (b) $y = \frac{4}{x-4}$

(c) $y = \frac{4x+5}{2x}$ (d) $y = \frac{-1}{x}$




6) (a) $y = \frac{x^2+4}{2}$ (b) $y - 5 = (x - 3)^2$

(c) $y = -x^2 - 6x - 5$ (d) $\sqrt{y} = x - 1$

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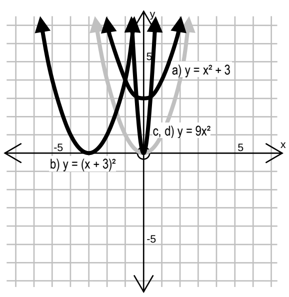
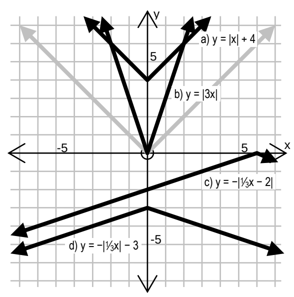
**SOLUTIONS**

A. (1) base graph: $y = x^2$

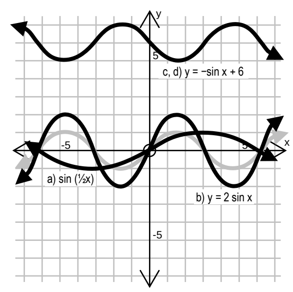



(c) and (d) are the same.

(2) base graph: $y = |x|$

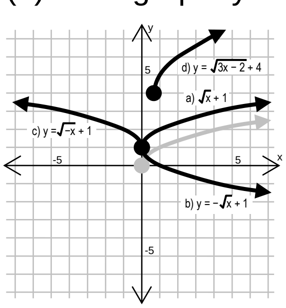



(3) base graph: $y = \sin x$

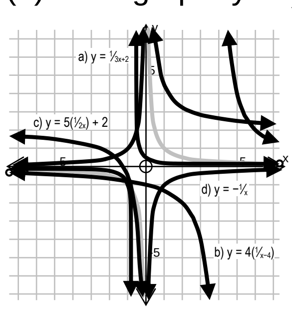



(c) and (d) are the same.

(4) base graph: $y = \sqrt{x}$




(5) base graph: $y = \frac{1}{x}$




(6) base graph: $y = x^2$