**Conic Sections:**

**The Hyperbola**

The equation for a hyperbola has both an $x^2$ and $y^2$ term, with one of them being added and the other subtracted. Once the equation is in standard form, which one is subtracted ($x^2$ or $y^2$) determines whether the hyperbola is “horizontal” or “vertical”.

**FORMULA FOR HYPERBOLAS**

Once the formula for the hyperbola is in standard form (described below), $a$ is always in the denominator of the term that’s added, and $b$ is always in the denominator of the term that’s subtracted.

**Horizontal Transverse Axis:**

\[
\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1
\]

centre: $(h, k)$

vertices: $(h + a, k), (h - a, k)$

foci: $(h + c, k), (h - c, k)$, where $c^2 = a^2 + b^2$

asymptotes: $y - k = \frac{\pm b}{a} (x - h)$

**Vertical Transverse Axis:**

\[
\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1
\]

centre: $(h, k)$

vertices: $(h, k + a), (h, k - a)$

foci: $(h, k + c), (h, k - c)$, where $c^2 = a^2 + b^2$

asymptotes: $y - k = \frac{\pm a}{b} (x - h)$

**Example 1:** Find the centre, vertices, foci and asymptotes of the hyperbola $x^2 + 8x - y^2 + 10y = 13$.

**Solution:** First we need to get the equation into the standard form. We start by completing the squares for $x$ and for $y$.

\[
\begin{align*}
(x^2 + 8x) - (y^2 - 10y) &= 13 \\
(x^2 + 8x + 16 - 16) - (y^2 - 10y + 25 - 25) &= 13 \\
(x^2 + 8x + 16) - 16 - (y^2 - 10y + 25) + 25 &= 13 \\
(x^2 + 8x + 16) - (y^2 - 10y + 25) &= 13 + 16 - 25 \\
(x + 4)^2 - (y - 5)^2 &= 4 \\
\frac{(x + 4)^2}{4} - \frac{(y - 5)^2}{4} &= 1
\end{align*}
\]

Now we can see $h, k, a$ and $b$: $h = -4, k = 5, a = 2$ and $b = 2$. The $x$ term is added, so its
denominator has \( a^2 \). This hyperbola has a horizontal transverse axis. The centre is at \((h, k)\), or \((-4, 5)\). The vertices are at \((h \pm a, k)\), or \((-2, 5)\) and \((-6, 5)\). We calculate \( c \):

\[
c^2 = a^2 + b^2 \\
= 2^2 + 2^2 = 8 \\
\therefore c = 2\sqrt{2}
\]

The foci, then, are at \((h \pm c, k)\), or \((-4 \pm 2\sqrt{2}, 5)\). The asymptotes are:

\[
y - k = \frac{\pm b}{a} (x - h) \\
y - 5 = \frac{\pm \sqrt{8}}{2} [x - (-4)] \\
y - 5 = \pm(x + 4) \\
y - 5 = x + 4 \text{ or } y - 5 = -x - 4 \\
y = x + 9 \text{ or } y = -x + 1
\]

**Example 2:** Find the equation of the hyperbola with vertices at \((5, 0)\) and \((-5, 0)\) and foci at \((6, 0)\) and \((-6, 0)\).

**Solution:** First, we must determine whether this hyperbola has a horizontal or a vertical transverse axis. The two vertices have the same \( y \) value, as do the foci, so we have a horizontal transverse axis. (Vertical transverse axes have the same \( x \) value for all four points.) The distance between the two vertices is equal to \( 2a \):

\[
2a = \sqrt{(5 - (-5))^2 + (0 - 0)^2} \\
= \sqrt{10^2 + 0^2} = 10 \\
\therefore a = 5
\]

The coordinate \((5, 0)\) is the one that’s farther to the right, so it must be \((h + a, k)\). This means \( k = 0 \), and \( h + a = 5 \). Since \( a = 5 \), \( h \) must be 0.

We can get \( b \) by calculating the distance between the centre and either focus, which is \( c \):

\[
c = \sqrt{[6 - 0]^2 + [0 - 0]^2} \\
= \sqrt{36 + 0} = 6 \\
c^2 = a^2 + b^2 \\
b^2 = c^2 - a^2 \\
= 36 - 25 = 11 \\
\therefore b = \sqrt{11}
\]

We have \( h, k, a \) and \( b \), so we can get the standard form of the equation of the hyperbola:

\[
\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1 \\
\frac{(x-0)^2}{5^2} - \frac{(y-0)^2}{\sqrt{11}^2} = 1 \\
\frac{x^2}{25} - \frac{y^2}{11} = 1
\]
Example 3: Find the equation of the hyperbola with vertices at \((4, -15)\) and \((4, 1)\) and asymptotes at \(y = 2x - 15\) and \(y = -2x - 1\).

Solution: First, we must determine whether this hyperbola has a horizontal or a vertical transverse axis. The two vertices have the same \(x\) value, so we have a vertical transverse axis. The distance between the two vertices is equal to \(2a\):

\[
2a = \sqrt{(4 - 4)^2 + [(-15) - 1]^2} = \sqrt{0^2 + 16^2} = 16
\]

\[
\therefore a = 8
\]

The vertex at \((4, 1)\) is the one that’s farther up, so it must be \((h, k + a)\). This means \(h = 4\), and \(k + a = 1\). Since \(a = 8\), \(h\) must be \(-7\).

Since \(a\) and \(b\) are distances, the equation for the asymptote with a positive coefficient on \(x\) must be of the form \(y - k = \frac{a}{b}(x - h)\). In fact, the coefficient on \(x\) must be \(\frac{b}{a}\):

\[
\frac{a}{b} = 2
\]

\[
\frac{8}{b} = 2
\]

\[
b = 4
\]

We have \(h, k, a\) and \(b\), so we can get the standard form of the equation of the hyperbola:

\[
\frac{(y - k)^2}{a^2} - \frac{(x - h)^2}{b^2} = 1
\]

\[
\frac{(y - 7)^2}{8^2} - \frac{(x - 4)^2}{4^2} = 1
\]

\[
\frac{(y + 1)^2}{4^2} - \frac{(x + 1)^2}{9} = 1
\]

\[
\frac{(y + 7)^2}{36} - \frac{(x - 4)^2}{16} = 1
\]

EXERCISES
A. Find the centre, vertices, foci and asymptotes for each hyperbola:

1) \(\frac{x^2}{4} - \frac{y^2}{9} = 1\)  
5) \(-x^2 + y^2 + 16y = 17\)

2) \(\frac{(y - 1)^2}{4} - \frac{(x + 1)^2}{9} = 1\)  
6) \(x^2 + 4x - y^2 + 8y = 3\)

3) \(x^2 - y^2 = 9\)  
7) \(x^2 + 2x - 4y^2 + 4y - 1 = 0\)

4) \(4x^2 - 4y^2 = 1\)

B. Find the equation of a hyperbola with the following features:

1) vertices: \((3, 0), (-3, 0)\); foci: \((4, 0), (-4, 0)\)

2) vertices: \((-1, 1), (-1, -3)\); foci: \((-1, 2), (-1, -4)\)

3) vertices: \((-4, 10), (-4, -2)\); asymptotes: \(y = 3x + 16, y = -3x - 8\)

C. Graph the hyperbola from Exercise A6, including the asymptotes.
SOLUTIONS

A:  
(1) ctr.: (0, 0); vert.: (2, 0), (−2, 0); foci: (±√13, 0); asym.: y = ±3/2 x
(2) ctr.: (−1, 1); vert.: (−1, −1), (−1, 3); foci: (−1, 1 ± √13); asym.: y = 2/3 x + 5/3, y = −2/3 x + 1/3
(3) ctr.: (0, 0); vert.: (−3, 0), (3, 0); foci: (±3√2, 0); asym.: y = x, y = −x
(4) ctr.: (0, 0); vert.: (−1/2, 0), (1/2, 0); foci: (±√2, 0); asym.: y = x, y = −x
(5) ctr.: (0, −8); vert.: (0, −17), (0, 1); foci: (0, −8 ± 9√2); asym.: y = x + 8, y = −x + 8
(6) ctr.: (−2, 4); vert.: (−2, 1), (−2, 7); foci: (−2, 4 ± 3√2); asym.: y = x + 6, y = −x + 2
(7) ctr.: (−1, 1/2); vert.: (−2, 1/2), (0, 1/2); foci: (−1 ± √5/2, 1/2); asym.: y = 1/2 x + 1, y = −1/2 x

B:  
(1) \frac{x^2}{9} - \frac{y^2}{7} = 1  
(2) \frac{(y+1)^2}{4} - \frac{(x+1)^2}{5} = 1  
(3) \frac{(y-4)^2}{36} - \frac{(x+4)^2}{4} = 1

C:  
[Note: Even though the y² term was subtracted in this question, this hyperbola is vertical!]

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