Lines
Slopes, Intercepts & Equations

SLOPE
\[ m = \frac{\text{rise}}{\text{run}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}, \]
- it's positive if the line rises to the right
- it's negative if the line falls to the right
- it's zero if the line is horizontal
- it's undefined if the line is vertical

SLOPE-INTERCEPT FORM
\[ y = mx + b \]
where \( m = \) slope of the line
\( b = y\)-intercept

POINT-SLOPE FORM
\[ y - y_1 = m(x - x_1) \]

STANDARD FORM
\[ Ax + By = C \]
\( A, B, C \) are integers; \( A > 0 \)

FINDING INTERCEPTS
A. X-INTERCEPT
   1. Set \( y = 0 \) in the equation.
   2. Solve for \( x \).
B. Y-INTERCEPT
   1. Set \( x = 0 \) in the equation.
   2. Solve for \( y \).

PARALLEL LINES
Non-vertical lines are parallel if and only if they have the same slope and different \( y \)-intercepts.
(All vertical lines are parallel.)

PERPENDICULAR LINES
Non-vertical lines are perpendicular to each other if and only if their slopes are negative reciprocals of each other.
This means the product of their slopes is \(-1\). If \( m \) is the slope of one line, then \(- \frac{1}{m}\) is the slope of the other.
(Vertical lines are perpendicular to horizontal lines.)

DISTANCE FORMULA
\[ d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \]

MIDPOINT FORMULA
coordinates are: \( \left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right) \)

VERTICAL & HORIZONTAL LINES
Vertical lines have equations of the form \( x = a \). They are parallel to the \( y \)-axis. All points on the line have the same first coordinate.
Horizontal lines have equations of the form \( y = b \). They are parallel to the \( x \)-axis. All points on the line have the same second coordinate.
Example 1: Express in standard form: y = \frac{5}{7}x + \frac{4}{7}.

Solution: Isolate the constant term and get the x and y terms on the other side.

\[ y = \frac{5}{7}x + \frac{4}{7} \]
\[ -\frac{5}{7}x + y = \frac{4}{7} \]

We need all the numbers in the problem to be integers, and the coefficient on x needs to be positive. We can multiply the entire equation by the LCD to get rid of the fractions and by -1 to get a positive coefficient for x:

\[ 7 \times (-\frac{5}{7}x + y) = 7 \times \frac{4}{7} \]
\[ -5x + 7y = 4 \]
\[ -1 \times (-5x + 7y) = -1 \times 4 \]
\[ 5x - 7y = -4 \]

There are several methods to find an equation of a line. All of them involve finding the slope first, if it hasn’t already been given to you.

Example 2: Find the equation of the line that has a slope of 5 and a y-intercept of 8.

Solution: We can use y = mx + b. \( m \) is the slope and \( b \) is the y-intercept.

\[ m = 5; \ b = 8 \]
\[ y = mx + b \]
\[ y = 5x + 8 \]

Example 3: Find the equation of the line that has a slope of 3 and contains the point (5, 4).

Solution: We can use the point-slope equation, \( y - y_1 = m(x - x_1) \). \( m \) is the slope and (\( x_1, y_1 \)) is the point.

\[ m = 3; \ (x_1, y_1) = (5, 4) \]
\[ y - y_1 = m(x - x_1) \]
\[ y - 4 = 3(x - 5) \]
\[ y - 4 = 3x - 15 \]
\[ y = 3x - 11 \]

Example 4: Find the equation of the line containing the point (4, 4) and (−6, −1).

Solution: This time we do not know the slope, so we must calculate it.

\[ m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-1 - 4}{-6 - 4} = \frac{-5}{-10} = \frac{1}{2} \]

Now that we have the slope, we can use either point to finish the equation.

\[ y - y_1 = m(x - x_1) \]
\[ y - (4) = \frac{1}{2}(x - 4) \]
\[ y - 4 = \frac{1}{2}x - 2 \]
\[ y = \frac{1}{2}x + 2 \]
Two lines are **parallel** if their slopes are the same, i.e. \( m_1 = m_2 \).

Two lines are **perpendicular** if the product of their slopes is \(-1\), i.e. \( m_1m_2 = -1 \). We also say that the slopes of perpendicular lines are **negative reciprocals** of each other, i.e. \( m_1 = \frac{-1}{m_2} \). The negative reciprocal of \( \frac{2}{3} \) is \( -\frac{3}{2} \), and the negative reciprocal of 5 is \( -\frac{1}{5} \).

**Example 5:** Find the equation of the line parallel to \( 3x + y = 5 \) containing the point \((1, -2)\).

**Solution:** The best way to find the slope is to convert to slope-intercept form:

\[
3x + y = 5 \\
y = -3x + 5
\]

It’s \( y = mx + b \), so \( m = -3 \). Now use the point-slope equation.

\[
y - y_1 = m(x - x_1) \\
y - (-2) = -3(x - (1)) \\
y + 2 = -3x + 3 \\
y = -3x + 1
\]

**Example 6:** Find the equation of the line perpendicular to \( 3x + y = 5 \) containing the point \((1, -2)\).

**Solution:** Again, we need to know the slope of the line we’ve been asked for. From the previous example, \( m = -3 \). The negative reciprocal of \(-3\) is \( \frac{1}{3} \), so \( \frac{1}{3} \) is the slope of any line perpendicular to \( 3x + y = 5 \).

\[
y - y_1 = m(x - x_1) \\
y - (-2) = \frac{1}{3}(x - (1)) \\
y + 2 = \frac{1}{3}x - \frac{1}{3} \\
y = \frac{1}{3}x - \frac{1}{3}
\]

**EXERCISES**

**A.** Write in slope-intercept form:

1) \( y - 3 = 2(x - 4) \)  
2) \( y + 2 = 3(x + 5) \)  
3) \( y - 6 = 2(x - 1) \)  
4) \( x - y = 4 \)  
5) \( 2x + 3y = 6 \)  
6) \( x - 4y = 7 \)

**B.** Write in standard form:

1) \( y = 3x + 6 \)  
2) \( y = -\frac{1}{2}x + \frac{5}{2} \)  
3) \( y = \frac{5}{6}x - \frac{2}{3} \)  
4) \( y - 7 = 4(x + 2) \)  
5) \( y - \frac{3}{4} = \frac{1}{2}(x + 5) \)  
6) \( y + 1 = -\frac{1}{7}(x - 3) \)

**C.** Find the x-intercept and y-intercept:

1) \( y = 2x + 6 \)  
2) \( y - 4 = 3(x + 5) \)  
3) \( 2x - 3y = -6 \)  
4) \( 4x + 6 = 3y - 2 \)

**D.** Find the distance between these pairs of points, and find the midpoint of the points:

1) \((3, 5) \text{ and } (-1, 7)\)  
2) \((4, 7) \text{ and } (0, -3)\)  
3) \((8, -1) \text{ and } (-4, 5)\)  
4) \((2, 6) \text{ and } (-1, 10)\)  
5) \((-6, \frac{1}{2}) \text{ and } (4, \frac{9}{2})\)  
6) \((5, 5) \text{ and } (-1, \frac{7}{2})\)
E. Find the equations of the lines with these slopes and y-intercepts:

1) \( m = 4, \ b = 3 \)  
2) \( m = -3, \ b = 0 \)  
3) \( \text{slope} = 0, \ y\text{-intercept} = -1 \)  
4) \( \text{slope} = -\frac{1}{2}, \ y\text{-intercept} = -5 \)  
5) \( \text{y-intercept} = 4, \ \text{slope} = 0 \)

F. Find the equations of the lines with these slopes, and that contain these points:

1) \( m = 5; \ (1, 3) \)  
2) \( m = -4; \ (3, -5) \)  
3) \( \text{slope} = -2, \ \text{point} = (-5, -4) \)  
4) \( \text{point} = (3, 2), \ \text{zero slope} \)  
5) \( \text{point} = (1, 1), \ \text{slope is} \ \frac{1}{3} \)  
6) \( \text{slope is equal to that of} \ y = 2x - 5; \ (3, 8) \)

G. Find the equations of the lines that contain these pairs of points:

1) \( (3, 5), (4, 0) \)  
2) \( (-1, 2), (3, 7) \)  
3) \( (2, 6), (0, -2) \)  
4) \( (4, 2), (6, 2) \)  
5) \( (1, 3), (1, 5) \)  
6) \( (1, 3), (1, 5) \)

H. Find the equations of the lines that are parallel to the given lines, and contain the given points:

1) \( y = 5x - 2, \ (3, 11) \)  
2) \( 2x + y = 5, \ (1, -2) \)  
3) \( 3x - 4y = 6, \ (2, 3) \)  
4) \( -x - 5y = 3, \ (-2, -4) \)  
5) \( 2x - 4y = -13 \)  
6) \( x + 7y = -4 \)

I. Find the equations of the lines that are perpendicular to the given lines, and contain the given points:

1) \( y = -2x - 5, \ (4, 6) \)  
2) \( 4x - 3y = 7, \ (8, -2) \)  
3) \( -2x + 5y = 13, \ (-5, 1) \)  
4) \( x + 7y = 10, \ (-1, 5) \)  
5) \( x - y = 3, \ (6, -5) \)  
6) \( x - y = 3, \ (6, -5) \)

**Solutions**

A. 

1) \( y = 2x - 5 \)  
2) \( y = 3x + 13 \)  
3) \( y = 2x + 4 \)  
4) \( y = x - 4 \)  
5) \( y = -\frac{2}{3}x + 2 \)  
6) \( y = \frac{1}{4}x - \frac{3}{4} \)

B. 

1) \( 3x - y = -6 \)  
2) \( x + 2y = 5 \)  
3) \( 5x - 6y = 4 \)  
4) \( 4x - y = -15 \)  
5) \( 2x - 4y = -13 \)  
6) \( x + 7y = -4 \)

C. 

1) \( \text{x-int:} -3, \ \text{y-int:} 6 \)  
2) \( \text{x-int:} 6 \)  
3) \( \text{x-int:} 19 \)  
4) \( \text{x-int:} -3, \ \text{y-int:} 2 \)  
5) \( \text{x-int:} -2, \ \text{y-int:} \frac{8}{3} \)

D. 

1) \( d = 2\sqrt{5}, \ \text{mdpt:} (1, 6) \)  
2) \( d = 2\sqrt{29}, \ \text{mdpt:} (2, 2) \)  
3) \( d = 6\sqrt{5}, \ \text{mdpt:} (2, 2) \)  
4) \( d = 5, \ \text{mdpt:} (\frac{1}{2}, 8) \)  
5) \( d = 2\sqrt{29}, \ \text{mdpt:} (-1, \frac{5}{2}) \)  
6) \( d = \frac{3}{2}\sqrt{17}, \ \text{mdpt:} (2, \frac{17}{4}) \)

E. 

1) \( y = 4x + 3 \)  
2) \( y = -3x \)  
3) \( y = -1 \)  
4) \( y = -\frac{1}{2}x - 5 \)  
5) \( y = 4 \)

F. 

1) \( y = 5x - 2 \)  
2) \( y = -4x + 7 \)  
3) \( y = -2x - 14 \)  
4) \( y = 2 \)  
5) \( y = \frac{1}{3}x + \frac{2}{3} \)  
6) \( y = 2x + 2 \)

G. 

1) \( y = -5x + 20 \)  
2) \( y = \frac{5}{4}x + \frac{13}{4} \)  
3) \( y = 4x - 2 \)  
4) \( y = 2 \)  
5) \( x = 1 \)

H. 

1) \( y = 5x - 4 \)  
2) \( y = -2x \)  
3) \( y = \frac{3}{4}x + \frac{3}{2} \)  
4) \( y = -\frac{1}{5}x - \frac{23}{5} \)  
5) \( y = 3 \)

I. 

1) \( y = \frac{1}{2}x + 4 \)  
2) \( y = -\frac{3}{4}x + 4 \)  
3) \( y = -\frac{5}{2}x - \frac{23}{2} \)  
4) \( y = 7x + 12 \)  
5) \( y = -x + 1 \)