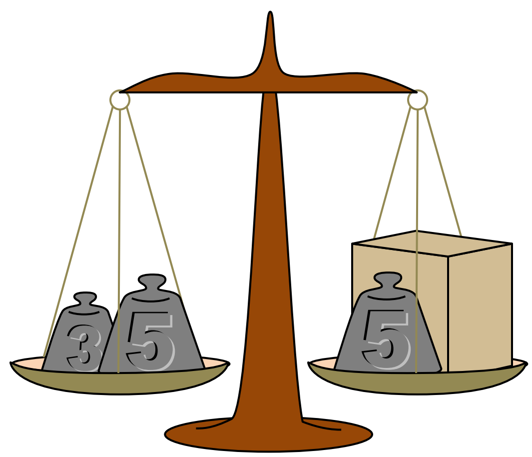
Fundamentals of Algebra

Algebra can be a scary subject because of the variables and all the new rules. This worksheet will explain how variables and the rules for algebra work.

VARIABLES

The letters that you see in algebra problems are **variables**, placeholders for numbers that we either don't know, or that we don't want to be specific about. They behave like pronouns in a sentence. If someone breaks into a house, we can talk about how *he* got in, *his* fingerprints and whether we catch *him*. The burglar is a specific person but we don't know his identity, so we use a generic pronoun. Similarly, we can say about anyone who gets arrested for burglary: *he* could go to jail for up to 5 years. There's no specific person here. We mean every person, or any person, who fits a particular description. Variables do the same thing. They're a "box" that we keep a number in.

SOLVING ALGEBRA PROBLEMS



Consider the following problem: You have a balance scale with a number of weights and an unknown weight inside a box. You want to know how much weight is in the box without having to open it.

Right now, we can see that the scale is balanced: there is the same amount of weight on both sides. However, because there is an extra weight on the pan with the box, we can't tell immediately how much weight is in the box. (This isn't a hard problem, but we're going to look at the process of solving this problem so that we can solve harder problems, and adapt the ideas to algebra. This is a useful problem-solving technique!) The best possible situation would be that the box is alone on one side, and only numerical weights are on the other. Then we would know that the box contained as much weight as there was on the other pan.

We can't just remove the 5-weight from the pan on the right. If we do, the scale will tip out of balance, and we won't know how much weight is in the box. If we add or remove weights, we need to keep the scale in balance. This suggests a rule for solving these problems: **whatever we do to one side, we must do exactly the same thing to the other side.** If we remove both 5-weights from the left and right pans, the scale will remain in balance. Then we're left with our best-case scenario: the only thing on the right pan is the box, and there are only number weights on the left. The box must contain a 3-weight.

Authored by Darren Rigby

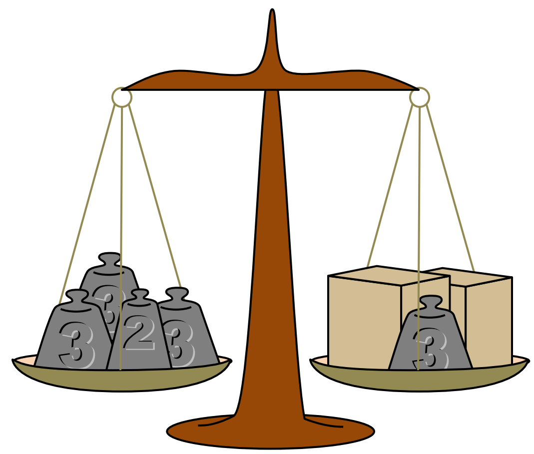
So how does the scale problem relate to algebra? We can write an algebra question that matches the scale problem like this:

$$8 = x + 5$$

We use a variable to represent the unknown weight inside the box. There's also a 5-weight on the pan with the box, so that pan weighs x + 5. The left pan we can see weighs 8. To solve the problem, we're going to get the box or "x" alone — we subtract the 5 from both sides of the equation:

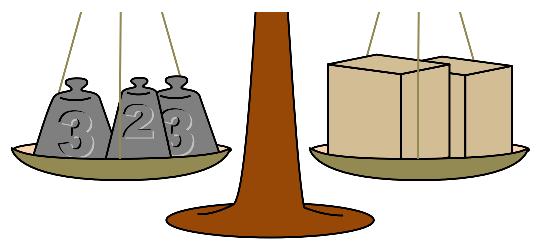
$$\begin{aligned} 8 &= x + 5 \\ 8 - 5 &= x + 5 - 5 \\ 3 &= x \end{aligned}$$

Of course, we get the same solution. You may have heard that you specifically want "x = …" and then a number. The variable can be on the right side as well. An algebra problem is just like the scale, it doesn't matter which weights are in which pan. Either the scales balance, or they don't.

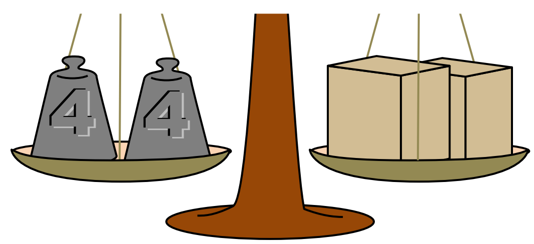


Here's another problem. We now have two boxes on one side of the scale. We are told that identical boxes have the same weight in them. (If they didn't, we could not solve the problem in this form; there isn't enough information.) Once again, the best way to figure out what's in a box is to move weights around so that one box is alone in one pan.

Our first move is to get rid of the 3-weight from the pan on the right, and we also remove a 3-weight from the pan on the left.



Now we have the problem of getting one of the boxes off the scale. Since we don't know how much it weighs, we can't take an equal weight from the other side. We can divide the weights on the left into two equal weights and then take one of those off. We'd be taking off as much weight as we leave behind. That will keep the scale in balance.



With the specific weights on the scale, it's not possible to divide them into two equal groups, but it would be possible if we replace the weights with two 4-weights. There's no reason why that shouldn't work; they represent the same amount of weight, so on the scale they do the same job. That's the other main rule of algebra: **it is always possible to replace part of the problem with something else that has the same value** without needing to change the other side at all.

Now we can take half the weight off of each pan, and we see that the box must contain a 4-weight.

The corresponding algebra problem looks like this:

$$11 = 2x + 3$$

We write “2x” meaning 2 x’s, or two times x. Writing a number in front of a variable indicates multiplication. We do this because “2 × x” would be confusing.

The solution to the problem would be:

$$\begin{aligned} 11 &= 2x + 3 \\ 11 - 3 &= 2x + 3 - 3 \\ 8 &= 2x \\ 8 \div 2 &= 2x \div 2 \\ 4 &= x \end{aligned}$$

This was a two-step solution: we subtracted and then we divided. Could we have done those steps in the other order? It’s possible, but remember the first rule: we have to do the same thing to both *sides*, not just the variable! Think of the scale again: If we had taken the original weights on the left and separated them into two piles, we wouldn’t really be taking half the weight from each pan — a box wouldn’t be half the weight in the right pan. We wouldn’t know how to divide the weights in the left pan. Similarly, this is wrong:

$$\begin{aligned} 11 &= 2x + 3 \\ 11 \div 2 &= 2x + 3 \div 2 \\ 5\tfrac{1}{2} &= x + 3 \\ 5\tfrac{1}{2} - 3 &= x - 3 \\ 2\tfrac{1}{2} &= x \end{aligned} \quad ✗$$

We broke one of the rules, so we don’t get the right answer. If we made a mistake like this, how could we tell? An easy way to check your answer in an algebra problem is to **plug in** your answer: replace the variable in the question with your answer and see if we have a correct equation. We can check 4 and 2½ as answers to this problem:

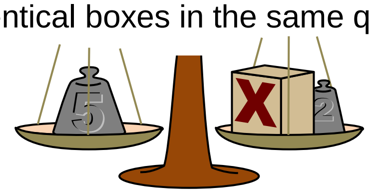
$$\begin{aligned} 11 &= 2x + 3 \\ 11 &= 2 \times \mathbf{4} + 3 \\ 11 &= 8 + 3 \\ 11 &= 11 \end{aligned} \quad ✓$$

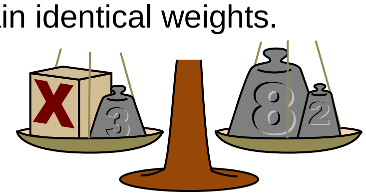
$$\begin{aligned} 11 &= 2x + 3 \\ 11 &= 2 \times \mathbf{2\tfrac{1}{2}} + 3 \\ 11 &= 5 + 3 \\ 11 &= 8 \end{aligned} \quad ✗$$

Now we can be sure 4 is the answer to the problem.

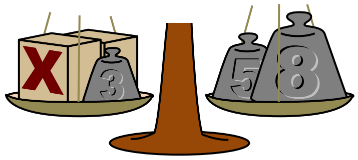
EXERCISES

A. Write the equations that correspond to the weights on these scales. You can assume that identical boxes in the same question contain identical weights.

1) 

2) 

(*A. Write the equations that correspond to the weights on these scales. You can assume identical boxes have identical weight.*)



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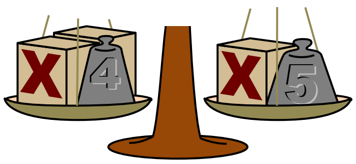

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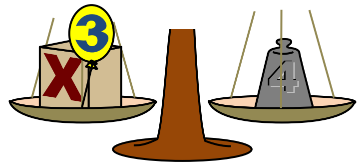
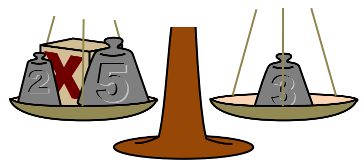


5)

(The helium balloon makes that pan *lighter* by 3.)



6)

B. Solve the algebra problems you created in A.

1)

2)

3)

4) [*Hint:* There are boxes on both pans. How can you make this problem simpler?]

5) [*Hint:* How can you make both sides *heavier*?]

6)

C. Check your solutions by plugging your answers into the questions.

SOLUTIONS

A: (1) 5 = x + 2 (2) x + 3 = 10 (3) 2x + 3 = 13 (4) 2x + 4 = x + 5 (5) x – 3 = 4 (6) x + 7 = 3

B: (1) x = 3 (2) x = 7 (3) x = 5 (4) Subtract x from both sides: x = 1 (5) Add 3 to both sides: x = 7 (6) Subtract 7 from both sides: x = −4. The box holds a balloon!