Solutions of Quadratic Equations

If a quadratic equation cannot be solved by the sum-product method—i.e. looking at the coefficients and factoring the quadratic by inspection—then we can use the **quadratic formula** to get the solutions. (The quadratic formula will work for all quadratic equations, but because it's complex, it's usually saved as a last resort.)

For a quadratic equation in the form $ax^2 + bx + c = 0$, the solutions are:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Example 1: Solve $x^2 + 4x + 2 = 0$

Solution: There aren't two numbers that add up to 4 and multiply to 2, so we use the quadratic formula. In this equation, a = 1, b = 4 and c = 2. We plug these values into the formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-4 \pm \sqrt{4^2 - 4 \cdot 1 \cdot 2}}{2 \cdot 1} = \frac{-4 \pm \sqrt{16 - 8}}{2}$$

$$= \frac{-4 \pm \sqrt{8}}{2} = \frac{-4 \pm 2\sqrt{2}}{2} = -2 \pm \sqrt{2}$$

So the two solutions are $-2 + \sqrt{2}$ and $-2 - \sqrt{2}$.

In some questions, you are not asked for the solutions themselves, but for the **nature** of the solutions. In other words, what sorts of solutions can we expect for this equation, and how many of them? To find this out we use the radicand from the quadratic formula, $b^2 - 4ac$, which mathematicians call the **determinant**. The answer we get when we evaluate the determinant for a quadratic equation tells us the nature of the solutions:

If the determinant is **positive** $b^2 - 4ac > 0$ then there are **two real solutions**.
If it's a perfect square, the solutions will be rational.
If it's not a perfect square, the solutions will be irrational.

If the determinant is **zero** $b^2 - 4ac = 0$ then there is **one real solution**.

If the determinant is **negative,** $b^2 - 4ac < 0$, then there are **two non-real solutions**.

Example 2: Determine the nature of the solutions of $3x^2 - 5x + 4 = 0$.

Solution: We look at the determinant: a = 3, b = −5 and c = 4.

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$$\begin{aligned} b^2 - 4ac &= (-5)^2 - 4 \cdot 3 \cdot 4 \\ &= 25 - 48 \\ &= -23 < 0 \end{aligned}$$

There are two non-real solutions.

Example 3: Determine the nature of the solutions of $x^2 + 6x + 5$.

Solution: a = 1, b = 6 and c = 5

$$\begin{aligned} b^2 - 4ac &= 6^2 - 4 \cdot 1 \cdot 5 \\ &= 36 - 20 \\ &= 16 \end{aligned}$$

There are two real solutions, and because 16 is a square number, the solutions must be rational.

EXERCISES

A. Determine the nature of the solutions of these quadratic equations:

1) $x^2 - 4x + 4 = 0$
2) $x^2 + 7x - 3 = 0$
3) $2x^2 - 2x + 3 = 0$
4) $-x^2 - 4x + 5 = 0$
5) $-3x^2 - 5x + 7 = 0$
6) $\frac{1}{2}x - 8x + 32 = 0$

B. Solve using the quadratic formula:

1) $x^2 - 7x + 5 = 0$
2) $x^2 - x - 4 = 0$
3) $3x^2 - x - 5 = 0$
4) $-5x^2 + 3x + 1 = 0$
5) $3x^2 - 5x = 0$
6) $4x^2 - 9 = 0$

SOLUTIONS

A. (1) one real sol'n (2) two real irrational sol'ns (3) two non-real sol'ns
(4) two real rational sol'ns (5) two real, irrational sol'ns (6) one real sol'n

B. (1) $\frac{7 \pm \sqrt{29}}{2}$ (2) $\frac{1 \pm \sqrt{17}}{2}$ (3) $\frac{1 \pm \sqrt{61}}{6}$ (4) $\frac{3 \pm \sqrt{29}}{10}$ (5) $\frac{5 \pm \sqrt{25}}{6} = 0$ or $\frac{5}{3}$ (6) $\pm \frac{\sqrt{144}}{8} = \pm\frac{3}{2}$