Simplifying Radicals

Like all mathematical expressions, we want to simplify radicals as much as possible so they're easy to understand, interpret, and visualize. There are two main ways to simplify radicals: reducing them, and resolving radical expressions with a mix of indices (which is the plural of "index").

The **index** of a radical is the little number on the front that tells us what kind of radical it is. An index of 3 indicates a cube root, of 4 a fourth root, and so on. We don't write the "2" for a square root, but that's still the index. Indices for radicals must be whole numbers greater than 1.

REDUCING RADICALS

To reduce a radical with an index of n, we must factor the radicand (the expression under the radical symbol) and look for factors that appear at least n times. (So if it's a cube root, the index is 3, and we need factors that are there at least 3 times.)

Example 1: Simplify: $\sqrt[3]{a^6b^7c^2}$

Solution: The laws of radicals let us break this up into several radicals multiplied together. (This isn't necessary to solve the problem, but it does help to demonstrate what's going on in the problem.)

We can write this as: $\sqrt[3]{a^6b^7c^2} = \sqrt[3]{a^6} \cdot \sqrt[3]{b^7} \cdot \sqrt[3]{c^2}$

The exponent of 6 on the a divides evenly by 3 to give 2. The 7 on the b gives and answer of 2 with a remainder of 1, and we can further separate that leftover b from that radical: $\sqrt[3]{a^6} \cdot \sqrt[3]{b^7} \cdot \sqrt[3]{c^2} = \sqrt[3]{(a^2)^3} \cdot \sqrt[3]{(b^2)^3} \cdot \sqrt[3]{b} \cdot \sqrt[3]{c^2}$

The first two radicals can be evaluated and the leftover radicals combined again: $\sqrt[3]{(a^2)^3} \cdot \sqrt[3]{(b^2)^3} \cdot \sqrt[3]{b} \cdot \sqrt[3]{c^2} = a^2b^2\sqrt[3]{bc^2}$

Again, this example includes way more steps than should be necessary, but it shows that each distinct factor can be treated separately in problems like these.

Example 2: Simplify: $\sqrt{6048m^5n^4}$

Solution: The number 6048 is not a square number. To reduce this radical, we'll need to break this number up into its prime factors so we can see what it's made of. Prime factorizations are unique — no matter how we do it, we'll always arrive at the

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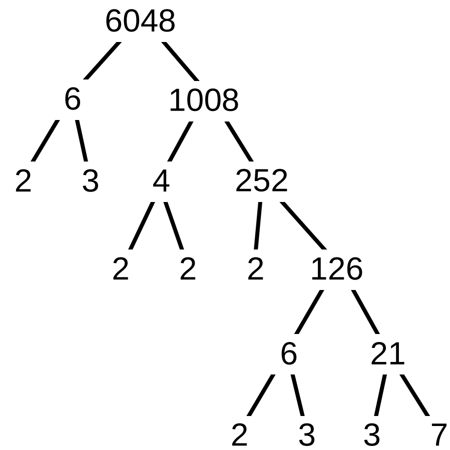
same breakdown. Factor trees are a good way to accomplish a prime factorization.

Here's a tip for factoring large numbers: if you can visually break the number up into pieces that are each divisible by the same number, then the big number is divisible by that number. The numbers 60 and 48 are both divisible by 6, so 6048 is divisible by 6. With a calculator, we see that $6048 = 6 \times 1008$. Both of these numbers can be broken down again. We know how to factor 6, and since 100 and 8 are both divisible by 4, 1008 is divisible by 4. We keep going, and we record this information in a tree diagram:



We now look at the "leaves" of the tree diagram — those numbers that are at the very ends of the branches, and those number should all be primes. We have five 2's, three 3's and a 7, so the prime factorization is $6048 = 2^5 \cdot 3^3 \cdot 7$.

The radical, then, can be written as $\sqrt{2^5 \cdot 3^3 \cdot 7 \cdot m^5 \cdot n^4}$ and we can finish this question like we did the last one:

$\sqrt{2^5 \cdot 3^3 \cdot 7 \cdot m^5 \cdot n^4} = 2^2 \cdot 3 \cdot m^2 \cdot n^2 \cdot \sqrt{2 \cdot 3 \cdot 7 \cdot m} = 12m^2n^2\sqrt{42m}$

ABSOLUTE VALUE SYMBOLS

The question $x^2 = 9$ is a quadratic equation, and so it has two solutions: 3 and −3. On the other hand we would like the symbol $\sqrt{\ }$ to be a function symbol, and therefore we want $x = \sqrt{9}$ to have just one solution. We call this solution, 3 (not −3), the **principal square root**.

We have similar definitions for any other root with an even index: $\sqrt[4]{x}$, $\sqrt[6]{x}$, and so on — negative numbers raised to even powers become positive, so we can't tell from x^4, x^6, etc., whether the original x was positive or negative. If the radicand also is being raised to an even power, then it doesn't matter. We know the answer will be positive.

We also know that there's no issue with odd powers, since negative numbers raised to odd powers remain negative, so $\sqrt[3]{x}$, $\sqrt[5]{x}$, etc., will retain the sign of x whether positive or negative.

All of this means that there's a tiny edge case where it matters whether x is positive or negative, but we must report a positive number only: ***If a variable in the question is raised to an even power, and we take an even-indexed root of that variable so that the answer is an odd exponent, that variable must have absolute value signs.*** This rule is so fiddly and specific that many questions give the instruction "Assume that no radicands for formed by raising negative numbers to even powers" or something similar. That instruction is telling you not to bother about the absolute value signs. If that instruction is not there, you must look for this special case!

MULTIPLE INDICES AND MIXED INDICES

Other problems require you to simplify the radicals themselves, rather than working on the radicands. In these cases, it's useful to convert radical notation into exponential notation by using fractions as exponents. Recall that $\sqrt[n]{x^m} = x^{m/n}$.

Example 3: Simplify: (a) $\sqrt[3]{x^2} \cdot \sqrt{x}$ (b) $\sqrt[5]{x^3 \sqrt[3]{x^2}}$

Solution: In both cases, we want to rewrite these expressions using exponents.

a) $\sqrt{x^2} \cdot \sqrt{x} = x^{2/3} \cdot x^{1/2}$

We're multiplying these expressions, so we add the exponents. We've made the exponents into fractions, so we need a common denominator:

$x^{2/3} \cdot x^{1/2} = x^{(2/3 + 1/2)} = x^{(4/6 + 3/6)} = x^{7/6}$

This is the answer as an improper fraction. If we make it a mixed number, we can easily see how it reduces:

$7/6 = 1\tfrac{1}{6}$, so $x^{7/6} = x^{(1\ 1/6)} = x^{(1 + 1/6)} = x^1 \cdot x^{1/6} = x\sqrt[6]{x}$

b) $\sqrt[5]{x^3 \sqrt[3]{x^2}} = (x^3 \cdot x^{2/3})^{1/5} = (x^{9/3} \cdot x^{2/3})^{1/5} = (x^{11/3})^{1/5}$

When we have two exponents on the same base like this, we multiply them:

$(x^{11/3})^{1/5} = x^{11/15} = \sqrt[15]{x^{11}}$

EXERCISES

A. Simplify. Assume that no negative numbers have been raised to even exponents in radicands.

1) $\sqrt{50}$
2) $\sqrt{162}$
3) $\sqrt{4800t^5}$
4) $\sqrt[3]{16r^5s^8}$
5) $\sqrt[3]{675a^{15}b^{19}c^2}$
6) $\sqrt[4]{243x^{11}y^{12}}$
7) $\sqrt[4]{2048m^{19}n^{25}p^3}$
8) $\sqrt[5]{28125h^{22}k^{26}}$
9) $\sqrt[6]{8064t^{13}w^4z^{27}}$

B. One and only one variable in section A should get absolute value signs if the instructions hadn't told you to ignore them. Which variable would be affected?

C. Simplify, expressing your answers with a single radical.

1) $\sqrt[3]{x^2} \cdot \sqrt[4]{x}$
2) $\sqrt[5]{y^3} \cdot \sqrt[6]{y} \cdot \sqrt{y}$
3) $\dfrac{\sqrt[3]{z^2}}{\sqrt{z}}$
4) $\dfrac{\sqrt[4]{a^3} \cdot \sqrt[6]{a}}{\sqrt[3]{a^2}}$
5) $\dfrac{\sqrt[8]{b^7}}{\sqrt[3]{b} \cdot \sqrt[12]{b^5}}$
6) $\sqrt{c^3 \sqrt[5]{c^2}}$
7) $\sqrt[3]{\sqrt[6]{\sqrt[4]{n^8}}}$
8) $\sqrt[3]{\sqrt[25]{p^{15}}\ \sqrt[18]{p^{12}}}$
9) $\sqrt[3]{q^7 + q^4}$

SOLUTIONS

A: (1) $5\sqrt{2}$ (2) $9\sqrt{2}$ (3) $40t^2\sqrt{3t}$ (4) $2rs^2\sqrt[3]{2r^2s^2}$ (5) $3a^5b^6\sqrt[3]{25b}$ (6) $3x^2y^3\sqrt[4]{3x^3}$ (7) $4m^4n^6\sqrt[4]{8m^3np^3}$ (8) $5h^4n^6\sqrt[5]{8m^3np^3}$ (9) $2t^2z^6\sqrt[6]{126tw^4z^3}$

B: The y^3 in A6 should be rendered either as $|y^3|$ or $y^2|y|$.

C: (1) $\sqrt[12]{x^{11}}$ (2) $y\sqrt[15]{y^4}$ (3) $\sqrt[6]{z}$ (4) $\sqrt[4]{a}$ (5) $\sqrt[8]{b}$ (6) $c\sqrt[10]{c^7}$ (7) $\sqrt[9]{n}$ (8) $\sqrt[45]{c^{19}}$ (9) $q\sqrt[3]{q^4 + q}$