Radicals

Radicals are sort of the opposites of exponents. The expression $\sqrt[3]{8}$ represents the number which must be multiplied 3 times to get an answer of 8, which in this case is 2. The parts of a radical expression have names:

INDEX ⇒ $\sqrt[5]{32}$ ⇐ RADICAND

For a **square root**—a radical with an index of 2—the two is usually not written. A radical with an index of 3 is a **cube root**, and radicals with an index of 4 or higher are fourth roots, fifth roots, and so on.

Radicals have laws that are similar to those for exponents:

Law:	*Examples:*
$x^{\frac{m}{n}} = \sqrt[n]{x^m} = (\sqrt[n]{x})^m$	$25^{\frac{3}{2}} = \sqrt[2]{25^3} = (\sqrt{25})^3 = 5^3 = 125$
$(\sqrt{x})^2 = x$ (when **x** is positive)	$(\sqrt{7})^2 = 7$
$\sqrt[\mathbf{n}]{\mathbf{x^n}} = \mathbf{x}$ (when **n** is odd) $\sqrt[\mathbf{n}]{\mathbf{x^n}} = \lvert\mathbf{x}\rvert$ (when **n** is even)	$\sqrt[3]{x^3} = x$; $\sqrt[5]{-32} = \sqrt[5]{(-2)^5} = -2$ $\sqrt{x^2} = \lvert x\rvert$; $\sqrt{x^4} = \lvert x^2\rvert = x^2$ **
**The absolute value sign is required for all variables in the answer that are raised to an odd power. The x in the answer to this second example is raised to an even power, therefore absolute value symbols are *not* required. (If the problem states that all variables are non-negative, then there is no need to worry about absolute value signs.)	
$\sqrt[\mathbf{n}]{\mathbf{xy}} = \sqrt[\mathbf{n}]{\mathbf{x}} \cdot \sqrt[\mathbf{n}]{\mathbf{y}}$	$\sqrt{9x^8} = \sqrt{9} \cdot \sqrt{x^8} = 3 \cdot x^{\frac{8}{2}} = 3x^4$
$\sqrt[\mathbf{n}]{\frac{\mathbf{x}}{\mathbf{y}}} = \frac{\sqrt[\mathbf{n}]{\mathbf{x}}}{\sqrt[\mathbf{n}]{\mathbf{y}}}$	$\sqrt[3]{\frac{27}{8}} = \frac{\sqrt[3]{27}}{\sqrt[3]{8}} = \frac{3}{2}$

ADDITION AND SUBTRACTION OF RADICALS

Only "like" radicals can be added or subtracted, i.e., radicals having the same index and the same radicand. When adding or subtracting, combine the coefficients and keep the "like" radical.

Example 1: Add: $6\sqrt[3]{2} + 10\sqrt[3]{2} - 5\sqrt[3]{2}$

Solution: $6\sqrt[3]{2} + 10\sqrt[3]{2} - 5\sqrt[3]{2} = (6+10-5)\sqrt[3]{2} = 11\sqrt[3]{2}$

Example 2: Simplify: $5 + 5\sqrt{3} + 3\sqrt{5}$

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Solution: Since none of the terms have like radicals (the radicands are different), this expression cannot be simplified any more than it already is.

SOLVING RADICAL EQUATIONS

When an unknown is inside a radical expression in an equation, we must isolate each radical and raise the entire equation to the appropriate power.

Example 3: Solve for x: $\sqrt{7x+5}-3=0$

Solution:

$$\begin{aligned} \sqrt{7x+5}-3 &= 0 \\ \sqrt{7x+5} &= 3 && \text{Isolate the radical.} \\ (\sqrt{7x+5})^2 &= 3^2 && \text{For a square root, square both sides.} \\ 7x+5 &= 9 \\ 7x &= 4 \\ x &= 4/7 \end{aligned}$$

For a problem of this type, we must check our solution in the original equation:

$$\sqrt{7x+5}-3 \overset{?}{=} 0$$

$$\text{L.H.S.} = \sqrt{7[4/7]+5}-3 = \sqrt{4+5}-3 = \sqrt{9}-3 = 3-3 = 0 = \text{R.H.S.} \checkmark$$

Example 4: Solve: $\sqrt{x+1}-\sqrt{x-2}=1$

Solution: We must isolate the radicals one at a time to get rid of them:

$$\begin{aligned} \sqrt{x+1}-\sqrt{x-2} &= 1 \\ \sqrt{x+1} &= 1+\sqrt{x-2} \\ (\sqrt{x+1})^2 &= (1+\sqrt{x-2})^2 \\ x+1 &= 1^2+2\sqrt{x-2}+(x-2) \\ 2 &= 2\sqrt{x-2} \\ 1 &= \sqrt{x-2} \\ 1^2 &= x-2 \\ 3 &= x \end{aligned}$$

We still have to check this answer:

$$\text{L.H.S.} = \sqrt{(3)+1}-\sqrt{(3)-2} = \sqrt{4}-\sqrt{1} = 2-1 = 1 = \text{R.H.S.} \checkmark$$

REDUCING RADICALS

Some radical expressions can be simplified by turning them into an expression with a smaller radicand and a coefficient in front. To do this, we need to look for a factor of the radicand that is a perfect square (or cube, or…). Then we can split the radical and simplify it.

Example 5: Simplify $\sqrt{8}$.

Solution: The square numbers are 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, and so on. Since 4 is a perfect square and a factor of 8, we can simplify this radical like this:

$$\sqrt{8} = \sqrt{4\times 2} = \sqrt{4}\times\sqrt{2} = 2\sqrt{2}$$

Example 6: Simplify $\sqrt[3]{48}$.
Solution: This time we have a cube root, so we need to examine perfect cubes. These are 1, 8, 27, 64, 125, 216, 343, and so on. This time we can use the factor of 8:

$$\sqrt[3]{48} = \sqrt[3]{8\times 6} = \sqrt[3]{8}\times\sqrt[3]{6} = 2\sqrt[3]{6}$$

RATIONALIZING THE DENOMINATOR

Rationalizing the denominator means rewriting a fractional expression to remove any radicals from the bottom. We do this by multiplying by 1.

Example 7: Rationalize the denominator of $\frac{4}{\sqrt{a}}$.

Solution: We multiply the top and bottom of the fraction by a radical which will cause the exponents on the radicand to add up to the index of the radical, or a multiple of it. Here the exponent is 1, and the index is 2, so:

$$\frac{4}{\sqrt{a}} = \frac{4}{\sqrt{a}}\times\frac{\sqrt{a}}{\sqrt{a}} = \frac{4\sqrt{a}}{\sqrt{a^2}} = \frac{4\sqrt{a}}{a}$$

Example 8: Rationalize the denominator of $\frac{4}{\sqrt[3]{a}}$.

Solution: Our method is the same, but this time the index is 3, so multiplying by $\sqrt[3]{a}$ isn't going to work. Multiplying by $\sqrt[3]{a^2}$ will:

$$\frac{4}{\sqrt[3]{a}} = \frac{4}{\sqrt[3]{a}}\times\frac{\sqrt[3]{a^2}}{\sqrt[3]{a^2}} = \frac{4\sqrt[3]{a^2}}{\sqrt[3]{a^3}} = \frac{4\sqrt[3]{a^2}}{a}$$

Example 9: Rationalize the denominator of $\frac{4}{\sqrt[3]{a^5}}$.

Solution: This time, we need to create a multiple of the index, since the exponent on the radicand is already higher than the index:

$$\frac{4}{\sqrt[3]{a^5}} = \frac{4}{\sqrt[3]{a^5}}\times\frac{\sqrt[3]{a}}{\sqrt[3]{a}} = \frac{4\sqrt[3]{a}}{\sqrt[3]{a^6}} = \frac{4\sqrt[3]{a}}{\sqrt[3]{(a^2)^3}} = \frac{4\sqrt[3]{a}}{a^2}$$

Example 10: Rationalize the denominator of $\frac{14}{4-\sqrt{2}}$.

Solution: This question is different from the others. Here we have a binomial denominator. To remove the radical in this denominator, we'll have to create a difference of squares. To do this, we multiply by the **conjugate** of the denominator, meaning, the denominator with the "−" replaced by "+", or vice versa. So the conjugate of $4-\sqrt{2}$ is $4+\sqrt{2}$.

$$\frac{14}{4-\sqrt{2}} = \frac{14}{4-\sqrt{2}}\times\frac{4+\sqrt{2}}{4+\sqrt{2}} = \frac{14(4+\sqrt{2})}{4^2-4\sqrt{2}+4\sqrt{2}-(\sqrt{2})^2} = \frac{14(4+\sqrt{2})}{16-2} = \frac{14(4+\sqrt{2})}{14} = 4+\sqrt{2}$$

OTHER NOTES

1) $\sqrt{x^2+9} \neq x+3$ and $(5 + a)^2 \neq 25 + a^2$. Binomial expressions like these need to be foiled out when they're squared. So: $\sqrt{x^2+6x+9} = x+3$ and $(5 + a)^2 = 25 + 10a + a^2$.

2) The symbol "$\sqrt{\ }$" means the **principal square root**, by which we mean the positive square root, so: if $x^2 = 4$, then $x = \pm 2$ (two possible answers), but $\sqrt{4} = 2$ (one answer, unless the question says otherwise).

3) You can *not* take the *even* root of a negative number ($\sqrt{-4}$ has no real solution), but you *can* take the *odd* root of a negative number ($\sqrt[3]{-8} = -2$).

4) If you square both sides of a radical equation, you must check your answer at the end to be sure you haven't picked up any extraneous solutions. If we solve $\sqrt{2x+5} = -3$, we would get the solution x = 2 (try it and see), but this doesn't satisfy the original equation since Note #2 tells us that $\sqrt{2x+5}$ needs to be a positive number.

EXERCISES

A. Simplify, if possible:

1) $2\sqrt{5} + 3\sqrt{5}$

2) $\sqrt{36} + 3\sqrt{16} - 2\sqrt{25}$

3) $\sqrt[3]{72} - \sqrt[3]{243}$

4) $\sqrt{12} + \sqrt{75} - \sqrt{6} - \sqrt{27}$

5) $\sqrt{24} + \sqrt{28} + \sqrt{216}$

6) $\sqrt[4]{144} - \sqrt{300}$

B. Solve:

1) $\sqrt{x+5} = 7$

2) $6 + 2\sqrt{x-1} = 13$

3) $3\sqrt{x+4} + 1 = 2x$

4) $\sqrt[3]{19-x} = 4$

5) $\sqrt{x+6} + \sqrt{x+1} = 5$

6) $\sqrt{2x+2} + \sqrt{x+2} = 7$

C. Rationalize the denominators:

1) $\dfrac{5}{\sqrt{2}}$

2) $\dfrac{3b}{\sqrt{6b}}$

3) $\dfrac{10}{3\sqrt{x+2}}$

4) $\dfrac{6x}{\sqrt[5]{x^3}}$

5) $\dfrac{1}{4+\sqrt{3}}$

6) $\dfrac{\sqrt{20}}{\sqrt{45} - \sqrt{28}}$

D. Simplify, if possible:

1) $\sqrt{x^2 - 16} + \sqrt{\frac{16}{9}}$

2) $\sqrt{3 - \sqrt{10 + \sqrt{36}}}$

3) $(6x + 7y)^2$

4) $\sqrt{9 - \sqrt[3]{729}}$

SOLUTIONS

A. (1) $5\sqrt{5}$ (2) 8 (3) $-\sqrt[3]{9}$ (4) $4\sqrt{3} - \sqrt{6}$ (5) $8\sqrt{6} + 2\sqrt{7}$ (6) $-8\sqrt{3}$

B. (1) x = 44 (2) x = 5¾ (3) x = 5 (4) x = −45 (5) x = 3 (6) x = 7

C. (1) $\frac{5\sqrt{2}}{2}$ (2) $\frac{\sqrt{6b}}{2}$ (3) $\frac{10\sqrt{x+2}}{3x+6}$ (4) $6\sqrt[5]{x^2}$ (5) $\frac{4-\sqrt{3}}{13}$ (6) $\frac{30+4\sqrt{35}}{17}$

D. (1) $\sqrt{x^2 - 16} + \frac{4}{3}$ (2) No real solution. (3) $36x^2 + 84xy + 49y^2$ (4) $\sqrt{0} = 0$