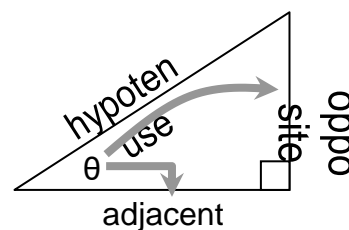# Essentials of Trigonometry 1

Usually in a geometry problem involving triangles, we have two pieces of information and we want a third piece. If we have two angles and want the third, we can use the fact that all three angles of a triangle add up to 180°. If we know two sides in a right triangle, and we want the third, we can use the Pythagorean theorem. If two of those three things are sides and one is an angle, then we must use trigonometry.

If I tell you that a right angle triangle has a 40° angle, that doesn't tell you what size the triangle is, but it does tell you the shape. All right triangles that have a 40° angle have the same shape, and their sides are proportional to each other. The triangle might be half as big as another, or twice as big, or ten times as big, but the ratio between the sides of the triangles will remain constant because the scaling factors will cancel out when we divide the lengths of the sides.

When taking the ratio of two of the sides, there are three different pairs of sides to choose from. To identify these pairs, we name the sides of the triangle. The longest side, which is always across from the right angle in the triangle is called the **hypotenuse**. For the other two sides, the name depends on which angle in the triangle we're looking at. When doing trigonometry with right triangles, we always do the calculations *with one of the two acute angles in mind*. If we're looking at the angle in the illustration above that's marked with a "θ" (theta), then the side at the bottom of the triangle is right next to that angle. In trigonometry, we say that it is adjacent to the angle, and it is called the **adjacent** side. The remaining side is across the triangle from angle θ. We say that it is opposite the angle, and so it is the **opposite** side. If we were working with the angle at the top of the triangle instead, the labels "adjacent" and "opposite" would switch places, but the hypotenuse wouldn't move. Which labels the two sides get in relation to the angle will tell you which tool to use to solve the problem. There are six possible ratios you can get from dividing two sides of a right triangle, and they all have names:

sine $\sin \theta = \dfrac{\text{opposite}}{\text{hypotenuse}}$ **cosine** $\cos \theta = \dfrac{\text{adjacent}}{\text{hypotenuse}}$ **tangent** $\tan \theta = \dfrac{\text{opposite}}{\text{adjacent}}$

cosecant $\csc \theta = \dfrac{\text{hypotenuse}}{\text{opposite}}$ **secant** $\sec \theta = \dfrac{\text{hypotenuse}}{\text{adjacent}}$ **cotangent** $\cot \theta = \dfrac{\text{adjacent}}{\text{opposite}}$

The first three functions, sine, cosine and tangent, all have buttons on your calculator while the other three don't. This is because for geometry, the last three are somewhat redundant — you can find them by calculating 1 ÷ sin, 1 ÷ cos and 1 ÷ tan. They are still necessary for calculus and other applications, so they should still be studied.

Authored by Darren Rigby

The order to divide the sides in the first three ratios can be memorized by remembering SOHCAHTOA: **S**ine **O**pposite/**H**ypotenuse, **C**osine **A**djacent/**H**ypotenuse, **T**angent **O**pposite/**A**djacent. You may hear people pronounce this as "soak-a-toe-uh".

*Example 1:*   Determine the length of the indicated side of the triangle.

Solution: First we need to decide which ratio to use. We know the angle and we know one side. That side is the adjacent. We need one side, and that side is the opposite side. Out of sin, cos and tan, the one that uses opposite and adjacent is tan. We can set up an equation to solve for the missing side:
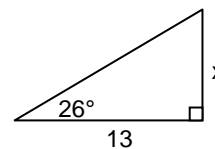
$$\tan \theta \ = \ \frac{\text{opposite}}{\text{adjacent}}$$

$$\tan 26° \ = \ \frac{x}{13}$$

$$13 \tan 26° \ = \ x$$

$$x \ = \ 13 \cdot 0.48773\ldots \approx 6.34$$

In general, a scientific calculator is needed to figure out the sin, cos or tan of most angles, but there are some that are capable of being worked out without a calculator. You'll be expected to memorize the information about those angles. They come from two triangles known as **special triangles**.

A square has 90° angles. If it is cut on the diagonal, that creates a triangle with two 45° angles. If we consider the side of the square to be 1 unit long, then we can use the Pythagorean theorem to determine that the hypotenuse of the triangle is $\sqrt{2}$ units long.
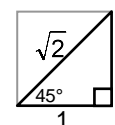
An equilateral triangle has 60° angles. If it's cut in half, that creates a right triangle with angles of 60° and 30°. If we consider the side of the equilateral triangle to have a length of 2 units, then the side that was cut in half is 1 unit long, and by the Pythagorean theorem, the altitude of the triangle is $\sqrt{3}$ units long. This triangle can be examined from either angle.
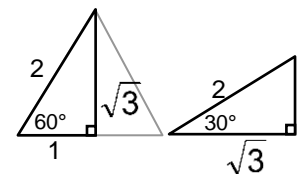
Example 2:   What is the exact length of the missing side of the triangle?

Solution: Because we see a 60° angle, we can use special triangles to determine the missing side exactly. We know the length of the side that is opposite the 60° angle, and we want to know the hypotenuse. The ratio that uses those two sides is sine. We set up a proportion:
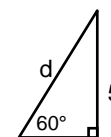
$$\sin \theta \ = \ \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\frac{\sqrt{3}}{2} \ = \ \frac{5}{d}$$

$$10 \ = \ \sqrt{3} \cdot d$$

$$d \ = \ 10 \div \sqrt{3} \ = \ \frac{10\sqrt{3}}{3}$$

The $\sqrt{3}/2$ from that problem can either be memorized from a table, or derived from the special triangle. The triangles are useful because they involve less memorization.

LABELLING TRIANGLES

In some problems, you see an angle labelled with a Greek letter such as θ. This is common practice in most of modern math and physics: angular things (including such physics concepts as angular velocity or angular momentum) get Greek letters, and linear things (including sides of triangles, linear velocity, regular momentum) get letters from the English alphabet.

In more classical geometry, points are always labelled with capital English letters, and sides within a triangle are named with the lowercase letter corresponding to the point <u>opposite</u> the side. So, if we are told that there is a triangle ABC, the only correct way to label the sides is shown here. Trig questions are often written with the assumption that you know about this naming convention.
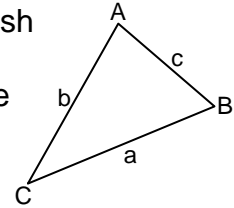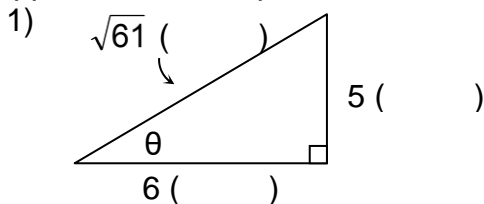
Angles in classical geometry are sometimes labelled from three points (instead of just one) to avoid ambiguities in complicated

diagrams. In the diagram at the right, there's only one ∠A, but there are many angles that might be called ∠C. Using the three-letter system, I find three points that are, in order, an endpoint, the corner and another endpoint. Angle ∠BCF is marked on the diagram.
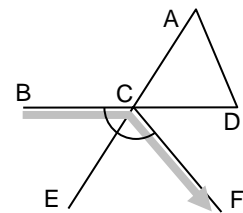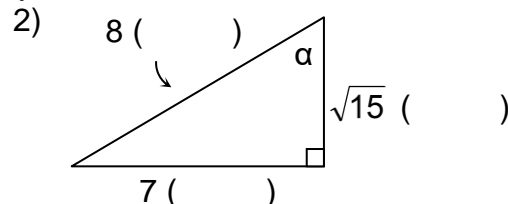
EXERCISES
A. In each triangle with a marked angle, label the three sides as adjacent, hypotenuse or opposite, and compute all six ratios exactly.
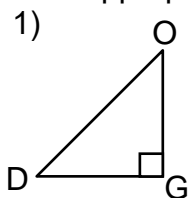
1) $\sqrt{61}$ (   )   5 (   )   6 (   )   θ

sin θ =   csc θ =
cos θ =   sec θ =
tan θ =   cot θ =

2)   8 (   )   α $\sqrt{15}$ (   )   7 (   )

csc α =   sec α =
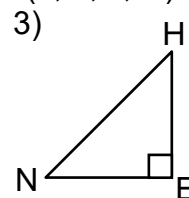cos α =   sin α =
cot α =   tan α =

B. For each triangle, express the given ratios in terms of the sides of the triangle labelled appropriately with lower case letters (a, b, c,…)
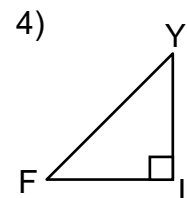
1)   O   D   G

sin O =
cos D =

2)   C   A   T

csc A =
sec C =

3)   H   N   E

tan N =
csc H =

4)   Y   F   L

cot F =
sin L =
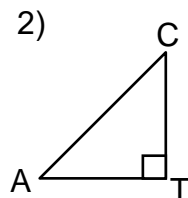
C. Solve for the indicated part of the triangles. Give exact values for sides.

1)

2)

3)

4)
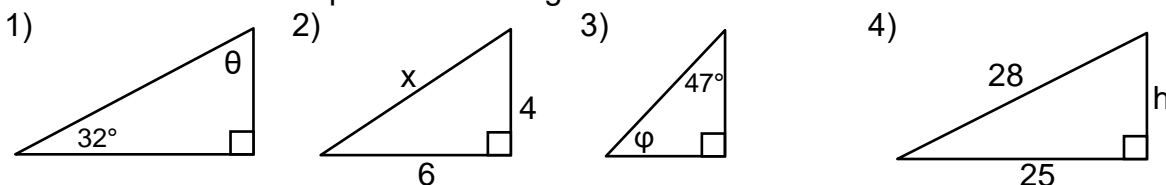
D. Identify which ratio (sin, cos or tan) is most appropriate for solving for the indicated sides of the triangles.
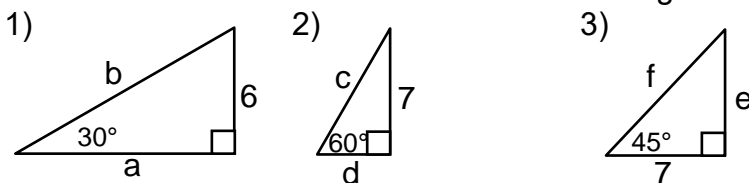
1)

2)

3)

4)
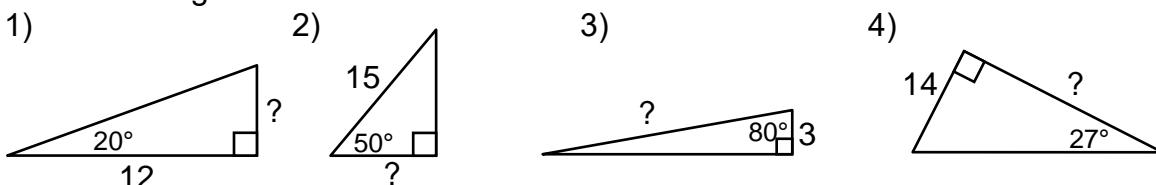
E. Solve for the indicated sides of the triangles in problems D1) – 4).

F. Find the measures of the unmarked angles and sides in D1) – 4).

G. Determine the lengths of the other two sides of these triangles. Which is the adjacent side and which is the opposite side?

1)

2)

3)

4)
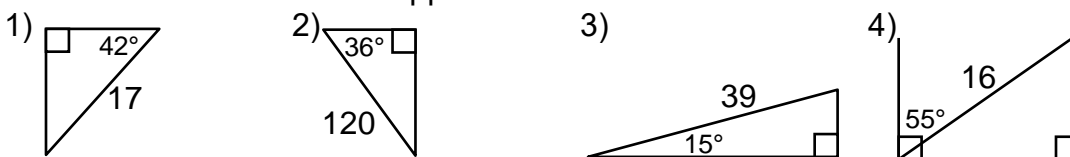
H. Solve for the two unknown sides of these triangles. Find exact values.

1)

2)

3)

SOLUTIONS

A. (1) $\sqrt{61}$: hyp, 5: opp, 6: adj; $\sin \theta = \frac{5\sqrt{61}}{61}$; $\cos \theta = \frac{6\sqrt{61}}{61}$; $\tan \theta = \frac{5}{6}$; $\csc \theta = \frac{\sqrt{61}}{5}$; $\sec \theta = \frac{\sqrt{61}}{6}$; $\cot \theta = \frac{6}{5}$. (2) 8: hyp, 7: opp, $\sqrt{15}$: adj; $\csc \alpha = \frac{8}{7}$; $\cos \alpha = \frac{\sqrt{15}}{8}$; $\cot \alpha = \frac{\sqrt{15}}{7}$; $\sec \alpha = \frac{8\sqrt{15}}{15}$; $\sin \alpha = \frac{7}{8}$; $\tan \alpha = \frac{7\sqrt{15}}{15}$

B. (1) $\sin O = \frac{9}{g}$; $\cos D = \frac{9}{g}$ (2) $\csc A = \frac{1}{a}$; $\sec C = \frac{1}{a}$ (3) $\tan N = \frac{n}{h}$; $\csc H = \frac{e}{h}$
   (4) $\cot F = \frac{f}{y}$; $\sin L$ – it is not possible to express this answer in terms of the other sides, since L is the right angle in the triangle.

C. (1) $\theta = 58°$ (2) $x = \sqrt{52} = 2\sqrt{13}$ (3) $\varphi = 43°$ (4) $h = \sqrt{159}$

D. (1) tan (2) cos (3) cos (4) tan

E. (1) 4.367… (2) 9.64… (3) 17.28… (4) 27.48…

F. (1) 70°, 12.77… (2) 40°, 11.49… (3) 10°, 17.01… (4) 63°, 30.84…

G. (1) horiz, adj.: 12.63, vert, opp: 11.38 (2) horiz, adj.: 97.08, vert, opp: 70.53
   (3) horiz, adj.: 37.67, vert, opp: 10.09 (4) horiz, adj.: 13.11, vert, opp: 9.18

H. (1) $a = 6\sqrt{3}$, $b = 12$ (2) $c = \frac{14\sqrt{3}}{3}$, $d = \frac{7\sqrt{3}}{3}$ (3) $e = 7$, $f = 7\sqrt{2}$