Properties of Real Numbers

To understand how algebra can be used to solve problems, we need to understand more about the sets of numbers we use, and the properties of the numbers in each one.

SETS OF NUMBERS

It helps to think of the historical context of different number systems. The way people used numbers has changed over time. It's only in the past century or so that we've begun to think of numbers without describing a quantity of things at the same time. "2 + 3 =" would not have been considered a practical problem; before doing the problem, most people would have asked, "Two what? Two apples? Two dollars?" They wouldn't think of numbers as an abstraction, only as a quantity of something they could hold or see.

Because of this, the oldest number system is the **natural numbers** or **counting numbers**. They were used to count things: three sheep, ten men, five years. The natural numbers represent quantities of things:

$$\{1, 2, 3, 4, 5, \ldots\}$$

In algebra, sets of numbers are represented by fancy "double-struck" letters. The natural numbers are represented by N, printed like this: $\mathbb{N}$, or hand-drawn like this: $\mathbb{N}$.

After the Dark Ages, a lot of lost mathematical and scientific information was re-introduced to Europe by the Moors. One mathematical concept they brought was the idea of nothing. If you had seven cattle and they all died, you had no cattle left, but before the 1200's this idea could not be expressed in numbers by most Europeans! When we include 0 with the natural numbers, we get **whole numbers**:

$$\mathbb{W} = \{\mathbf{0}, 1, 2, 3, 4, 5, \ldots\}$$

As banking and commerce increased, the next concept that changed the face of numbering systems was debt. If a merchant had promised to pay more money than he had, then it suddenly became very important to keep track of how much money the merchant owed. The amount that such a person owes is expressed as a **negative number**. When we put the negative numbers together with the whole numbers, we get **integers**. The integers are represented by the letter $\mathbb{Z}$:

$$\mathbb{Z} = \{\mathbf{\ldots -5, -4, -3, -2, -1,}\ 0, 1, 2, 3, 4, 5, \ldots\}$$

Why Z, rather than I? Z stands for the German name for integers: *die Zahlen*. The letter I is used for a set of numbers that you'll learn about later called imaginary numbers, which are numbers that are not real numbers.

Authored by Darren Rigby

For our next set of numbers, we go much further back in time for the concept of sharing. If two or more people are supposed to get parts of something, the ability to divide something equally becomes important. As a practical problem, you'd have a countable quantity of items to share and a number of people to share between. These numbers would be divided, but the answer would not always be a whole number. The simplest way to write the answer to a problem like this would be to use the numbers involved: If I have to divide three jugs of olive oil among seven customers, then each customer gets $\frac{3}{7}$ of a jug of olive oil. **Fractions** were invented. The set of **rational numbers**, $\mathbb{Q}$, is defined as the set of all possible solutions to problems involving division of integers. Mathematically:

$$\mathbb{Q} = \{x \mid x = \frac{p}{q}, \text{ where p and q are integers}\}$$

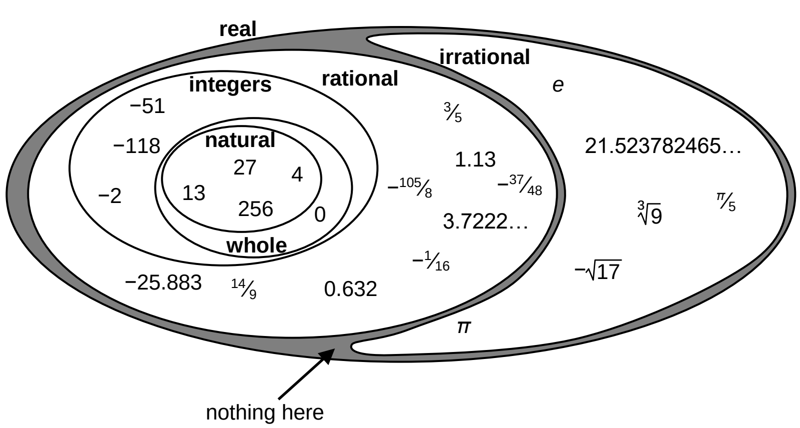
The Q stands for "quotient", which is the name for the answer to a division problem. The word *rational* in this case means "related to a ratio". A ratio is another way of describing a division problem.

Expressed as a decimal fraction, any rational number either **terminates** (or stops, e.g., $\frac{1}{8} = 0.125$) or **repeats** itself endlessly (e.g., $\frac{3}{7} = 0.428571428571428571\ldots$). Any terminating or repeating decimal number must be rational. All integers are rational numbers as well since any integer, z, can be expressed as the fraction $\frac{z}{1}$.

The last set of numbers describes any distance you can draw. If I draw a square with a side that is 1 unit long, how far apart are the opposite corners of the square? If I draw a circle with a diameter of 1 unit, what is the circumference of the circle? These are some very simple problems whose answers are not rational. The set of numbers that can represent any distance in reality is known as the **real numbers**, $\mathbb{R}$ or ℝ. The real numbers aren't defined as a set like the rational numbers, but as any point on the **number line**, or more properly, as any distance between "0" on the number line and any other point on the line (since the concept of real numbers comes from distances).

All rational numbers are real numbers, of course. The other real numbers that are not rational, are called **irrational numbers**, and they're represented by $\overline{\mathbb{Q}}$. The bar is a symbol in sets that means "not", since irrational numbers are literally "not rational numbers". The numbers $\sqrt{2}$ and π (the distance across the square and the distance around the circle) are common examples of irrational numbers. The decimal equivalents of irrational numbers go on endlessly, but never fully repeat. The decimal value of $\sqrt{2}$ is 1.4142135623731… and π is 3.141592653589832….

We can draw these sets of numbers as a Venn diagram:



PROPERTIES OF NUMBERS

The properties you need to know for all real numbers and the operations they apply to are in the list below. In these definitions, r, s and t can be any real number.

Identity — There is an **additive identity,** a number that when added to a real number doesn't change its value. The additive identity is 0: $r + 0 = r$.

There is a **multiplicative identity**, a number that when multiplied by a real number doesn't change its value. The multiplicative identity is 1: $r \times 1 = r$.

Inverses — Every number has an **additive inverse** (or **opposite**), a number that when added to a real number gives you the additive identity (0). The additive inverse of r is −r: $r + (-r) = 0$.

Almost every number has a **multiplicative inverse** (or **reciprocal**): a number that when multiplied by a real number gives you the multiplicative identity (1). The only real number that does not have a reciprocal is 0. The multiplicative inverse of r ($r \neq 0$) is $1/r$: $r \times 1/r = 1$. For a fraction r/s ($r, s \neq 0$), the reciprocal is s/r.

Commutative Law — When adding numbers, order doesn't matter: $r + s = s + r$.

When multiplying numbers, order doesn't matter: $r \times s = s \times r$.

Associative Law — The way numbers are bracketed when they're added doesn't matter: $(r + s) + t = r + (s + t)$.

The way numbers are bracketed when they're multiplied doesn't matter: $(r \times s) \times t = r \times (s \times t)$.

Distributive Law — Multiplying a real number by a sum yields the same result as multiplying each term individually and adding. You should recognize this as collecting like terms. $r \times (s + t) = (r \times s) + (r \times t)$. This is called the distributive law for multiplication over addition.

Multiplying a real number by a difference yields the same result as multiplying each term individually and subtracting. $r \times (s - t) = (r \times s) - (r \times t)$. This is called the distributive law for multiplication over subtraction.

Example 1: Use the commutative and associative laws to write $(3 + 5) + 4$ three other ways.

Solution: The commutative law applies to the two numbers in brackets: $(5 + 3) + 4$. It also applies to the bracketed sum and the last term: $4 + (3 + 5)$. The associative law applies to the whole expression: $3 + (5 + 4)$. There are many other solutions.

EXERCISES

A. From the following set of numbers, identify all the (1) whole numbers, (2) rational numbers, (3) integers, (4) real numbers, (5) natural numbers, (6) irrational numbers: $\{3/1, \sqrt{39}, 7, -28, 0.257, \pi/2, 0, 0.1248163264\ldots, 5.8717171\ldots, 13/0, \sqrt[3]{125}, 24/7, \sqrt{-2}\}$

B. Write the additive inverse for these numbers.

1) 3	3) −77	5) 0
2) 62	4) −23	6) ¼

C. Write the multiplicative inverse for these numbers.

1) 3
2) −5
3) $\frac{1}{41}$
4) $-\frac{1}{110}$
5) $\frac{4}{9}$
6) 0.333333….

D. Rewrite the following expressions using the associative and commutative laws.

1) 10 + (3 + 2)
2) (a + b) + c
3) (2 × 5) × 7
4) (5 × 15) × 25

E. Simplify using the distributive law.

1) 4 × (5 + b)
2) 11 × (a − 3)
3) 4y + 2y
4) 12n − 4n + 3n

F. The properties for real numbers were listed for addition and multiplication. Could there be similar properties for subtraction among real numbers? If so, write the equation that demonstrates the property. If not, write a counterexample.

1) property of identity
2) property of inverses
3) commutative law
4) associative law

G. Another property that numbers can have is **closure**. If a set is closed under an operation, then using the operation on two members of the set will always give a member of the set. Addition is closed for the real numbers; if you add two reals, the answer is always a real. Division is not closed under the integers; it is possible to divide two integers and get an answer that is not an integer (1 ÷ 2 = ½).

Decide whether these sets are closed under the given operations. If they are not, give an example to prove it.

1) real numbers, subtraction
2) whole numbers, multiplication
3) natural numbers, subtraction
4) rational numbers, division
5) even integers = {…−2, 0, 2, 4, …}, addition
6) odd integers = {…−3, −1, 1, 3, …}, addition
7) whole numbers, taking an average
8) irrational numbers, addition

SOLUTIONS

A: (1) $\{\frac{3}{1}, 7, 0, \sqrt[3]{125}\}$ (2) $\{\frac{3}{1}, 7, -28, 0.257, 0, 5.8717171\ldots, \sqrt[3]{125}, \frac{24}{7}\}$ (3) $\{\frac{3}{1}, 7, -28, 0, \sqrt[3]{125}\}$ (4) $\{\frac{3}{1}, \sqrt{39}, 7, -28, 0.257, \frac{\pi}{2}, 0, 0.1248163264\ldots, 5.8717171\ldots, \sqrt[3]{125}, \frac{24}{7}\}$ (5) $\{\frac{3}{1}, 7, \sqrt[3]{125}\}$ (6) $\{\sqrt{39}, \frac{\pi}{2}, 0.1248163264\ldots\}$

B: (1) −3 (2) −62 (3) 77 (4) 23 (5) 0 (6) $-\frac{1}{4}$

C: (1) $\frac{1}{3}$ (2) $-\frac{1}{5}$ (3) 41 (4) −110 (5) $\frac{9}{4}$ (6) 0.3333… = $\frac{1}{3}$, so the reciprocal is 3.

D: Many answers possible: (1) (10 + 3) + 2 (2) (b + a) + c (3) 7 × (2 × 5) (4) 15 × (25 × 5)

E: (1) (4 × 5) + (4 × b) = 20 + 4b (2) (11 × a) − (11 × 3) = 11a − 33
(3) y × (4 + 2) = y × 6 = 6y (4) n × (12 − 4) + 3n = (n × 8) + (n × 3) = n × (8 + 3) = 11n

F: (1) r − 0 = r (2) The subtractive inverse of r is r: r − r = 0
(3) does not hold: 3 − 2 ≠ 2 − 3 (4) does not hold: (10 − 5) − 3 ≠ 10 − (5 − 3)

G: (1) closed (2) closed (3) not closed: 3 − 7 is not natural (4) not closed: $\frac{3}{0}$ is not rational (5) closed (6) not closed: the sum of any two odd numbers is even (7) not closed: $\frac{3+4}{2}$ is not a whole number. (8) not closed: since irrational numbers are real, each one has an additive inverse: π + (−π) is not irrational.